ACHILLES AND THE TORTOISE

This, gentlemen, is a stiff dose of philosophical logic. This is, moreover, a three-part article. But there's one simple reason why this magazine should carry it; it is a rigorously logical attack on the problem of inter-stellar flight. It is an integrated attack on the problem of what that fine old "space-warp" or "hyper-space" means in specific physical-science terms. And it starts, necessarily, with the ancient question "What is motion?" If we are to escape Relativity limitations, we must start at a more fundamental level, not build up from Relativity.

– part 1 of 3 –

by GOTTHARD GÜNTHER

It is probable that the problem of interplanetary space travel may be solved in scientific and technical terms with which we are already familiar. But the situation looks very different when we approach the question of interstellar or even intergalactic voyages. The distances involved are so gigantic that interplanetary methods will be useless, and so will be the logical and physical concepts on which they are based. The following article tries to present some novel concepts – implied by recent developments in mathematics and physical science – which seem to contain the solution to the problem how to cross Deep Space.

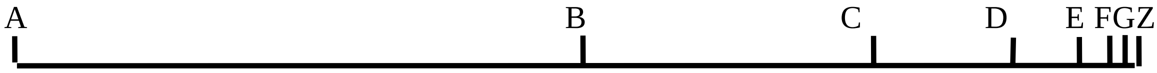
The ideas expressed in this article have grown out of a correspondence between John W. Campbell, Jr. and the author. Although Mr. Campbell has occasionally been referred to in the text, his share in the ideas expressed herein has been much greater. In fact there is hardly any side to the problem in the development of which he did not participate. So what he claims as his own should be recognized as such. The actual presentation, of course, is the exclusive responsibility of the author, and any errors that might have occurred are solely his.

So you really know precisely what motion in space means? When you get behind the steering wheel of your car and travel from, let us say, New York City to San Francisco, you know exactly what you do? Well, let us admit that in a practical sort of way you should indeed know what you are doing. If not, then may God help the other road users. It is, however, a very different matter if we ask the question: do we know in exact theoretical and scientific terms what motion in space actually is and how it happens? The answer is a most emphatic No! It may seem strange that something so commonplace, something we do every day as long as we live, involves unsolved logical and scientific problems. But that is the case. It is still a complete mystery to us, what actually happens when a physical body moves from one point in space to some other point.

There is a reason for it. Nowadays we are finally beginning to know what Matter is, and what basic laws seem to define its ultimate structure. But we have not the slightest idea what Space – the mere absence of anything "physical" – might mean. It stands to reason: as long as we cannot give even an approximate meaning to the general term "Space", it will be absolutely impossible to have even the haziest concept about what really does happen when a body moves in space from one point to another. If we try to explain it, we get entangled in contradictions and paradoxes – a clear indication that our present thinking methods are inadequate even to pose the problem.

This has been known since the times of ancient Greece, and the most famous exposition of the riddle that is offered by the phenomenon of motion in space is Zeno of Elea's paradox of

Achilles and the Tortoise. Achilles, the fastest runner that ever lived, cannot overtake the Tortoise, the slowest animal. Zeno's argument runs as follows: Let AZ be the race track.



Achilles starts from A, the Tortoise at the same time from B. If we assume that the Homeric hero runs twice as fast as the animal, the inference seems inevitable that both racers should reach point Z at the same time; but such was not Zeno's conclusion. This famous philosopher argued that while Achilles covers the distance AB, the Tortoise reaches point C. That is halfway between B and Z. When Achilles arrives at C, the animal must have reached D, this time midway between C and D. When Achilles is at D, that Tortoise must have gone to E. When Achilles passes E, the animal is necessarily at F. And when our hero is at F the Tortoise has again passed half the distance between F and Z and is, therefore, now at G, and so ad infinitum. It follows, so Zeno concludes, that Achilles can never overtake the Tortoise. And, incidentally, neither of the runners can ever reach Z.

The point of the argument is, of course, the influx of Time. Whenever Achilles reaches a designated point it has taken him time to get there from the preceding one, and during this time lapse the Tortoise has moved on to the next – as the animal is in constant motion. And no matter how short the distance will ultimately become, some time must always elapse till Achilles may cover it, and during that time interval the Tortoise shall have moved away from the point the pursuer is about to reach.

There is only one possibility that Achilles may catch up with the Tortoise. If the Homeric hero would move with infinite speed and would, therefore, cover the distances between A, B, C, D, … , Z in zero time, then the Tortoise would have no time to get away from point B as soon as the race starts. The beginning of the race and its finish would be the same identical moment. In other words: there would be no race at all. But if there is a race – with finite speeds for the racers – no overtaking could ever take place.

We all know from the practical viewpoint of our everyday experience that Zeno's argument is sheer nonsense. But the baffling thing is, although it is contradicted by the commonest actions in everybody's life, Zeno's point is *logically* unassailable. There is no technical flaw in his reasoning. He has indeed with his paradox touched t h e very problem of space and its relation to motion, and his argument indicates one of the deepest insights into the metaphysical structure of the world. Alfred North Whitehead once remarked: "I am fond of pointing out to my pupils that to be refuted in every century after you have written is the acme of triumph. I always make that remark in connection with Zeno. No one has ever touched Zeno without refuting him, and every century thinks it worth while to refute him." ("Essays in Science and Philosophy", New York 1947, p.114.) Obviously none of these refutations has ever been final. Zeno's paradox is now more than two thousand years old and the discussion about its merits is still going strong.

However, in modern textbooks on logic and metaphysics the reader may frequently find a (mistaken) statement to the effect that the infinitesimal calculus has finally solved Zeno's problem. The argument usually runs as follows: mathematically speaking the paradox of motion in space resolves itself into a problem of limits. If our race track AZ = x, and Achilles' handicap AB = 1, then obviously we have $x = 2$.

But Zeno constituted an unending series AB + BC + CD + DE + EF + … with the stipulation that every given distance (except the first) is exactly one half of the preceding one. We, therefore, obtain for x the equation:

$$x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots ad\text{-}infinitum$$

The question now is: does the right side of the equation equal 2? If that is the case, then it is mathematically demonstrated that Achilles must overtake the Tortoise at point Z.

This is the moment when the concept of "limit" is introduced. It is a device which enables us to circumscribe the infinity of numerical terms which turn up at the right side of our equation for x by a finite operation. The procedure involved was developed by the French mathematician Cauchy (1789-1857), and it shows that the right side of our equation is exactly 2. The "trick" of Cauchy's technique is, of course, that he eliminates the Infinite, because we cannot think it, and replaces it with the notion "limit." This substitute term permits a rigorous mathematical treatment [1] and produces a "solution" for Zeno's paradox.

Now let us have a closer look at the "solution". What does it really demonstrate? It proves in the most rigorous way that our common sense experience, that Achilles *does* overtake the Tortoise, is right. But we know that anyhow. And no person in his right senses ever doubted it. Not even Zeno himself! When he developed his famous paradox he meant something very different. Explaining it to his disciples he might have said: There is a trivial and everyday occurrence like motion in space. It is absolutely convincing to our senses, and nobody doubts its existence. Yet at the same time that trivial fact defies any attempt to *think* it in non-contradictory terms. The phenomenon of motion as such is positively unthinkable. And it is unthinkable because the concept of Infinity is involved. Think Infinity and you will have solved my paradox. But there is the rub: Infinity is unthinkable for human thought.

This shows us that the solution provided by modern calculus is no genuine logical solution. It detours the real difficulty, the problem of the Infinite, and replaces it by a different concept, the limit, which corresponds to our normal practical experience. It does not show us how to *think* the element of Infinity involved in the mystery of motion. It does exactly the opposite: it shows us how to avoid it. Zeno's original problem: what is it that creates an apparently insolvable paradox in our consciousness if we try to think "motion", has not been solved by the technique of calculus. This has been borne out by the history of calculus itself. Its first discoverers, Newton and Leibniz, tried very hard to exclude the concept of an infinitely small quantity from their mathematical procedures. However, they did not fully succeed. The validity of their methods was doubted, and the technique of calculus did not attain its full scientific rigor till the notion of Infinity was finally eliminated, and replaced by the more modest concept of "limit". The calculus was recognized to be incompetent to deal with problems of genuine Infinity and the very essence of Zeno's paradox remained unsolved.[2]

"So what?" the reader might ask. "Isn't it enough that we have a mathematical tool that can compute any sort of motion we might observe in Space? Computation is sufficient for all practical purposes. So why bother about mysterious metaphysical properties of Space, Time and Motion !"

I am sorry, but it is not as simple as all that. First, our computation methods are already insufficient, when we encounter a so-called three-body-problem. And then: what about interstellar travel? Existing and computable types of locomotion may be satisfactory for travel on this planet and even in interplanetary space. But they are decidedly unsatisfactory when it comes to bridging interstellar distances within reasonable time intervals. As long as we don't know anything about the structural properties of the so-called "continua"– Space *per se* is a "continuum" and so is Time – we cannot even ask the question whether these as yet unknown properties might permit types of locomotion, as yet equally unknown, by which a body (a spaceship) might alter its own position in space. The idea of a "space warp", so frequently encountered in science fiction points in that direction. If something like it existed, it would be a "motion" measured in terms of something other than distance – or time-units. The only way to discover and explore physical possibilities that might lie in this direction is the analysis of the paradox properties of the Space- and Time-continuum. And these properties are structural characteristics of the Infinite. Therefore Zeno should be very much alive with us.

It is obvious that the failure to make Zeno's problem disappear was the failure of mathematics to develop a method to deal with the Infinite. The Infinite was just the limit of our numerical conceptions. We could approach, but never reach it, and within its realm all operational procedures broke down. Infinite plus one was Infinite. Infinite plus a million was Infinite, and Infinite plus Infinite still was nothing but Infinite. In other words: Infinite was the absolute limit for the counting process and, therefore, the limit-concept of quantity in general. This was what children learned in school, and it was also the limit of wisdom for the accomplished mathematician.

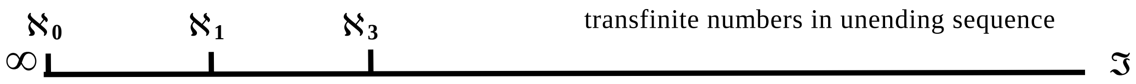
All this was changed, overnight so to speak, by the work of one man who ranks equal with the greatest in the history of mathematics. His name was Georg Cantor (1845-1918). He was born in Russia, lived the greater part of his life in Germany, and died as professor of mathematics at the University of Halle (Germany). During the final quarter of the last century Cantor published a series of articles which completely revolutionized our concept of number, of counting, and generally of quantity. In these articles Cantor transcended the concept of limit, thus ultimately bringing the very concept of Infinity within the grasp of mathematical technique.

Cantor's results are so startling, nay, so unbelievable and fantastic for the normal mind that they were first attacked by mathematicians from every quarter. Today they are basically accepted, and they have led to a most radical overhauling of the foundations of traditional logic as well as classical mathematics. What Cantor has discovered can be summarized by the following statement: We are in error if we assume that the process of counting is limited by the concept of the Infinite – and that Infinite itself has no definite quantitative properties. It is, on the contrary, possible to count *beyond* Infinite and to construct an unending series of numbers, the smallest of which is our traditional concept of Infinite. Any subsequent number in this series is of higher arithmetical magnitude than mere Infinite. Cantor called these numbers which are of higher numerical power than the mere limit-concept of Infinite the transfinite numbers, or transfinite sets.

In other words: Cantor distinguishes two structurally – different types of numbers. The first group are our familiar numbers which designate finite objects and relations.

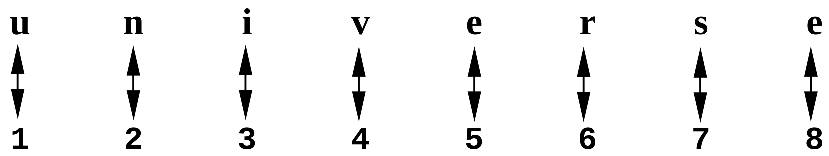
Finite realm of counting:

The second group comprises the Transfinite realm of counting:



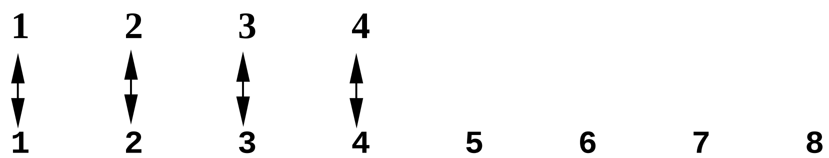
In order to designate these transfinite numbers Cantor used the first letter of the Hebrew alphabet, Aleph, with a numerical index. His numbers are, therefore, called the Aleph-numbers, or short: the Alephs. They begin, as our figure shows, with $\aleph_0$, or Aleph zero (sometimes called Aleph naught), which is the *completed* traditional Infinite, and ascend from there to higher and higher numerical powers of the Transfinite and finally converge against a transfinite limit $\mathfrak{I}_i$ [3]. In order to understand the fantastic orders of magnitude which are implied in the transfinite realm of counting one should realize that, if $\aleph_0$ is the completed traditional order of Infinite, then $\aleph_1$ is a number which represents the infinite power to the traditional infinite. Now, if a mathematician makes the unheard of claim that he has discovered a new type of number series by dint of which he can determine differences of magnitude within the Infinite he is, of course, under obligation to explain how his new concept of number differs from our familiar finite numbers. Cantor's explanation is very simple and can be understood without any specific mathematical training. We have to make only a few preliminary steps.

First we ask, what, do we do when we count? The answer is: we establish a one-to-one correspondence between a group of objects and a second group of numerical concepts. Let us, for instance, count the letters in the word "universe":



Our double-headed arrows indicate the one-to-one correspondence between letter and number, and we see that the *cardinal number* which determines the quantity of letters used in "universe" is eight. In other words "8" represents a set of integers which is equivalent to the number of letters in our word. Of course it is obvious, if we call "8" a set of integers, then "1", "2", "3", "4", …, are sub-sets of our original set. And as we have counted letters or could count apples, horses, cars or ideas, we might as well count any such sub-set.

Let us take for instance the sub-set "4" and count it with our original set:



Everybody can see that the result is not a one-to-one correspondence between our counting numbers (below) and our counted objects (set "4"), but a one-to-two correspondence. There is no equivalence between our counting numbers and the counted set. We discover here the basic logical characteristic of all finite numbers which can be expressed as follows:

No finite number set is equivalent to a proper sub-set of itself.

This maxim holds unconditionally for the finite realm of counting and its application tells us that a certain number in question is finite. But what is obvious for the Finite is false for the Infinite!

In order to understand what follows, please let me remind you that the numerical magnitude of a counted set is always established by a one-to-one correspondence with a counting set – as it was the case with the letters of "universe" and the set "8." Now turning to infinite sets it would seem that the set of all positive integers (even *and* odd) should be of higher numerical magnitude than the set of all even numbers. We again apply our system of pairing the counted (above) and the counting numbers:

1	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	…
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	…

No matter how long we continue our pairing – and we assume it to be an unending series – we shall never run out of counting numbers, but we shall also never exhaust the series of even numbers which we want to count. Of course, the class of even integers is "thinned out" as compared with the class of all integers, but this, "thinning out" has not the slightest effect on the order of numerical magnitude for the "thinned out" series. Such is the nature of even the lowest form of infinity!

To drive this most important point home I shall give you two more examples, of such "thinning out" processes and their one-to-one correspondence with our unending series of integers:

1	**8**	**27**	**64**	**125**	**216**	…
↕	↕	↕	↕	↕	↕	
1	**2**	**3**	**4**	**5**	**6**	…

The counted series in this case are the cubes (1^3, 2^3, 3^3, …) of the integers – and again we shall never run out of counted numbers as little as of counting integers. Both series are of equal numerical magnitude, because both converge against the same infinite limit.

As our last example we might finally stipulate that only such numbers shall be counted which begin with a "1", an odd number of zeros following:

10	**1000**	**100000**	**1000000**	…
↕	↕	↕	↕	
1	**2**	**3**	**4**	…

No matter how radical our "thinning out" process is, the unending series above our double-headed arrows can never be exhausted by our counting integers. In other words, there are "as many" numbers in the series 10, 1000, 100000, …, as there are in 1, 2, 3, … This seems to be the height of absurdity, but it is the inevitable logical consequence of the process we applied when we counted the letters in the word "universe".

It is now possible to state exactly what we mean if we call a set of numbers infinite. We defined a finite set as one which is not equivalent to a proper sub-set of itself. And we now say:

Any set that is equivalent to a proper sub-set of itself is infinite.

And this Infinite is the first number of Cantor's set of transfinite Alephs. It is the $\aleph_0$ of the transfinite realm of counting.

The next problem, of course, is how to proceed to our next transfinite number which should be "bigger" than our traditional Infinite. It is not too difficult to do so. Before we even begin to construct the next transfinite Aleph we can deduce what basic logical property it should have. The following table of properties, common and not-common to finite and infinite sets, should help:

type of set	special characteristic	common characteristic
finite	non-equivalent to sub-sets	denumerable
infinite	Equivalent to sub-sets	denumerable

This table shows the logical situation at one single look. Finite and infinite sets differ as to their equivalence characteristics, but they are both *denumerable*. That means, there is always a method of counting the members of the different sets. And the method is the same. It stands to reason, if we want to find a third type of numerical set which differs from the finite as well as the infinite, this third type will have to negate what is *common* to its predecessors. To word it positively, the next transfinite number will have to be *non-denumerable*.

To obtain an expression for a non-denumerable Aleph let us do some transfinite arithmetic. It bears, as you will see, very little resemblance to that of the finite numbers:

Addition: $\aleph_0 + 1 = \aleph_0$

$\aleph_0 + \aleph_0 = \aleph_0$

Multiplication: $2 \cdot \aleph_0 = \aleph_0$

$n \cdot \aleph_0 = \aleph_0$

(where n represents any finite number)

Also: $(\aleph_0)^2 = \aleph_0 \cdot \aleph_0 = \aleph_0$

$(\aleph_0)^n = \aleph_0$

There seems to be no variation in the result of these operations, but it is all very deceptive and treacherous. Because the result is quite different when we try:

$$(\aleph_0)^{\aleph_0}$$

This equation creates a new transfinite number of higher numerical magnitude than the first number of Cantor's series. This second Aleph number is non-denumerable.

But what does non-denumerability actually mean? We shall find out by following some of Cantor's trends of thought. Common sense tells us that there are more fractions than integers; for in between any two integers there is an infinite number of fractions. Alas – common sense is amidst alien corn in the land of the Infinite.

Although the rational fractions have no definite neighbors Cantor discovered a simple but elegant method to count them, thus proving their denumerability. He arranged the set of all rational fractions not in order of increasing magnitude (that is impossible) , but in order of ascending numerators and denominators in the following array:

1/1	**1/2**	**1/3**	**1/4**	**1/5**	...
2/1	**2/2**	**2/3**	**2/4**	**2/5**	...
3/1	**3/2**	**3/3**	**3/4**	**3/5**	...
4/1	**4/2**	**4/3**	**4/4**	**4/5**	...
5/1	**5/2**	**5/3**	**5/4**	**5/5**	...
...	...	...	...	...	...
...	...	...	...	...	...

Now the familiar one-to-one correspondence with the integers – necessary for the process of counting – may be effected.

1/1	**2/1**	**1/2**	**1/3**	**2/2**	**3/1**	**4/1**	**3/2**	**2/3**	...
↕	↕	↕	↕	↕	↕	↕	↕	↕	
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	...

It follows, the number of all rational fractions is denumerable, hence also of the order of magnitude of $\aleph_0$. It may be hard to believe that there are "only" as many rational fractions as there are integers, especially in view of the fact that there are an infinite number of fractions between any two integers, but such are the mathematics of the infinite. Even with adding all rational fractions to our previous concept of $\aleph_0$ we have not yet left the arithmetical dimension of denumerability. However, Cantor's greatest triumph came when he could show that the class of rational plus irrational numbers – i.e. of the so-called real numbers – is of a higher order of magnitude than the denumerable $\aleph_0$. The class of real numbers is non-denumerable.

His proof is based on a *reductio ad absurdum.* He assumed that the real numbers between 0 and 1 were countable and could be paired with the integers. All real numbers can be expressed as non-terminating decimals, and Cantor wrote them down in the following array for counting:

1↔0.	a_1	a_2	a_3	a_4	a_5	a_6	.	.	.
2↔0.	b_1	b_2	b_3	b_4	b_5	b_6	.	.	.
3↔0.	c_1	c_2	c_3	c_4	c_5	c_6	.	.	.
4↔0.	d_1	d_2	d_3	d_4	d_5	d_6	.	.	.
5↔0.	e_1	e_2	e_3	e_4	e_5	e_6	.	.	.
6↔0.	f_1	f_2	f_3	f_4	f_5	f_6	.	.	.
.	.	.	.	.	.	.	.	.	.

If this array (where a_1, a_2, …; b_1, b_2, …; et cetera, are ciphers of the series 0, 1, 2, …, 8, 9) is unending in horizontal as well as in vertical direction, it should contain *all* real numbers and thus be denumerable. But just the opposite is the case. This array exhibits the very con-tra-diction to the arguments that all sets are denumerable. For, no matter how our non-terminating decimals are actually arranged, it is always possible to find an infinity of other decimals which are not present in the array … *although the same is infinite.* The question is: how can we determine such omitted decimals? One might argue: as this set-up is unending and we can actually count only a finite number of individual decimals, the ones we claim to be omitted might still turn up in the as yet uncounted infinite reaches of our array. Cantor countered this argument by the discovery of his famous "diagonal procedure". This technique permits us to show that there are real numbers which are not represented in our arrangement.

Please take another look at the array of the, decimals. You will find that the ciphers a_1, b_2, c_3, d_4, e_5, f_6, … , are connected by a diagonal line. If we now construct a second decimal fraction (with Greek letters): α_1, β_2, γ_3, δ_4, ε_5, φ_6, … , where α_1 differs from a_1, β_2 differs from b_2, γ_3 differs from c_3 and so on, then this new number differs from all the unending decimal fractions in our original set in just one place. It differs from the first number in our table in the first place, from the second number in the second place, from the third number in the third place and generally from any n-th number in the n-th place – no matter how far toward the Infinite we may place n. It is, therefore, impossible that our "diagonal number" may turn up in the as yet unexplored reaches of our array. The diagonal number is, therefore, a real number between 0 and 1 *not* contained in our denumerable array.

It is further possible to repeat this procedure an infinite number of times by starting with a_2, a_3, a_4, … or b_1, c_1, d_1, … thus creating not only one but an infinite series of "diagonal numbers". It is herewith demonstrated that the set of all real numbers between 0 and 1 is non-denumerable and, therefore, of higher arithmetical magnitude than the denumerable Infinite. And since the same can be demonstrated for the set of all real numbers between 1 and 2, between 2 and 3, between 3 and 4, and so forth, it follows that the set of all real numbers is also non-denumerable.

If this is the case, we must for the first time in the history of human thinking admit logical distinctions within the realm of the Infinite. Because at least two different types of the Infinite have been determined: the denumerable order of the Infinite and the non-denumerable order which is of higher arithmetical power. As the set of the real numbers designates the so-called continuum, Cantor called the transfinite number which represents this set: the transfinite cardinal number c. By some further proof Cantor could show that the cardinal number of all univocal real functions, f, is even of higher transfinite order than c. Thus we already possess three Alephs. The Aleph of the classical Infinite which produced Zeno's paradox, the Aleph of the continuum and the Aleph of the univocal real functions.

The arithmetic of c is very much the same as that of $\aleph_0$. It is interesting and significant that when c is combined with $\aleph_0$ it swallows the latter completely. Thus we have: [4]

$$c + \aleph_0 = c$$
$$c - \aleph_0 = c$$
$$c \cdot \aleph_0 = c$$
$$c \cdot c = c$$

Even the following equation holds:

$$c^{\aleph_0} = c$$

But the result is different in the next case:

$$c^c = f$$

We pointed out before that the number of transfinite Alephs is infinite. Thus f is by no means the highest numerical concept we may conceive. We do know that there exists an unending series of higher Alephs that converges towards "Campbell's limit", because it can be proven that the set S of the sub-classes of any given class C always possesses a higher cardinal number than C. That means in the case of our number f that the set of its sub-sets is of higher transfinite order than f itself.[5] This process can be continued ad infinitum. So much about Cantor's daring creation.

The next installment of my article will show that Cantor's theory of the transfinite cardinal number c, the Aleph of the continuum, provides a genuine solution to Zeno's paradox.

More than that, Cantor's deductions provide an access to an entirely new concept of Space and to a mathematical basis for interstellar space travel. Some of the conclusions about the structural properties of Space, Time and Matter to be drawn from the theory of the Alephs are so startling and so absolutely beyond our present thinking habits that I cannot resist the temptation to finish this article with the statement, that interstellar travel will be possible the very day Achilles *really* overtakes the Tortoise. So far he does so only within an independent context of Nature – and against the anguished protest of our too limited powers of thinking. But the day of interstellar space travel will be here when Achilles overtakes the Tortoise in our thoughts as well as in Nature. In other words, when we have unraveled the secret of motion. The two following installments of this article have been written with the intent to bring this day nearer.

Footnotes:

[1] :

For the mathematically advanced reader some hints of the limit procedure are given. Our original equation for x is first reduced to the generalized expression

$x_n = 2 - \frac{1}{2^{n-1}}$ It is then shown that $\frac{1}{2^{n-1}} < E$

(where E is the smallest given number) reaches its "limit" only for $n \to \infty$

We therefore obtain $n \xrightarrow{\lim} \infty$ and $\frac{1}{2^{n-1}} = 0$. If we insert this value into our generalized expression we derive

$n \xrightarrow{\lim} \infty$ and $2 - 0 = 2$

That means: x has the value 2.

[2] :

Zeno's paradox deals with the notion of the spatial "continuum". And it was finally proved by Weierstrass in his example of a "continuous function" that the limit methods of calculus are insufficient to deal with the problem of actual infinity. (K. Weierstrass, "Erstes Beispiel einer stetigen, nirgends differenzierbaren Funktion", Journal für Mathematik, IX, 1875)

[3] :

For the idea that there should be an ultra-transfinite limit to all Alephs I am indebted to John W. Campbell. In a letter to me, of July 7, 1953, he calls it the transfinite number of the non-denumerable space of *imagination"*. In deference to his interesting ideas on this subject I have given our symbol for the transfinite limit the Index "i". The symbol $\mathfrak{I}$ stands for the second letter of the Hebrew alphabet: Beth.

[4] :

Aleph naught is usually represented in transfinite arithmetic by the letter, a. Mathematicians are in the habit of writing: c + a = c, etc. I have retained $\aleph_0$ in order to indicate that the Aleph order of the denumerable Infinite is established but not the transfinite magnitude of c.

[5] :

The principle of this transfinite induction can be illustrated by a very elementary example: Let the set A contain only the three integers 1,2,3. The cardinal number of our set is 3. However, the sub-sets of A (including the null-class) are (0), (1), (2), (3), (1,2), (1,3), (2,3), (1,2,3). Count them and you will find that the cardinal number of the set S of all the sub-sets is 8.

TO BE CONTINUED

ACHILLES AND THE TORTOISE

Part Two of Three Parts Relativity shows that nothing can move in space faster than light. The interstellar cruiser becomes possible only if we can, somehow, transcend that limitation. And one way may be simply that of asking, and finding a new answer, to the deeply fundamental question, "What do you mean by 'motion'?"

– part 2 of 3 –

by GOTTHARD GÜNTHER

The solution of a riddle or of a paradox is usually something in the nature of an anticlimax. The suspension is released, and the feeling of exciting wondermemt has passed. A famous. example is the story of the Gordian knot. When Alexander the Great conquered Asia he found in the city of Gordius an old chariot its yoke fastened with a cord that was tied in a complicated knot. The cord was so artfully twisted that no one had ever been able to loosen it. Moreover, there was an oracular Prophecy that he, who first would untie the knot, should rule Asia. Alexander tried, but he, too, failed. So he took his sword and "untied" the knot by cutting it in two.

I heard this story first, as a small boy in school. But even then I had an uneasy feeling. This solution seemed to me rather untidy – hardly more than a fraud. Sure, I was given the usual interpretation of that famous incident: There are problems the intellect cannot solve and only too often Man arrives at a puzzling impasse where only daring action can find a way out. This explanation did not satisfy me at all! Who said in the first place that the Gordian knot was untyable in the precise sense of the word? Well, no one ever did. The story only tells us that many tried, but nobody succeeded. That only proves the candidates were never good enough. And look at Alexander himself! He "solved" the problem in a manner of speaking, and got his Asiatic empire. But it was the most short-lived empire in the history of Man. It fell apart the day he died. It seems his method of untying the knot was only his private solution, not valid for anybody else.

The conventional solution of Zeno's paradox is just about in the same category I pointed out in Part I that the riddle of Infinity, involved in the problem of motion, seems to be insoluble. But mathematicians discovered the infinitesimal calculus, i.e. a special procedure which permitted us to abandon the material concept of actual Infinity. They replaced Infinity by the operational idea of the limit. This opened for the very first time a way to demonstrate in an exact mathematical manner what every child has already learned by countless experiences: that the fast runner always overtakes the slow runner. Practically speaking, the new infinitesimal method was of paramount importance. Modern technique and industry simply could not have been developed without the limit procedure. But hardly anything was gained as far as the precise theoretical concept of motion was concerned. It remained the mystery of old and defied all attempts to analyze it in rigid logical terms.

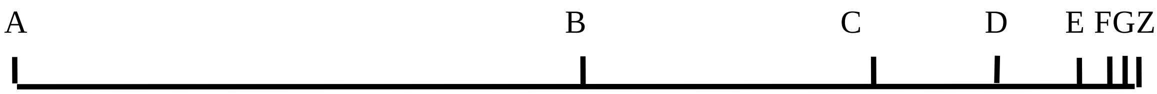
At this point an intellectually healthy and normal person is very much tempted to say: "So what if we do not understand motion? Newton's and Leibniz's calculus permits us to *use* it at will. And this is all that really, counts. What else do you want?" Exactly, what else could we want? Well, what about interstellar space travel? We surely want that! But the first step toward it is to understand that all known forms of locomotion are utterly and absurdly use-less, when we face the problem: how to traverse interstellar distances! Therefore, our big question is: Are there any other as yet unknown and structurally different forms of motion which are neither exemplified in our daily lives nor in the motion of the planets of a solar system in their orbits?

It is absolutely impossible to answer these and related questions satisfactorily unless we have a precise rational understanding of what motion really is. That means, unless we have positively succeeded in solving the problem of Zeno … instead of detouring it by eliminating its crucial element of actual Infinity. The infinitesimal calculus only demonstrated in definite mathematical terms that this mysterious X, called motion, is possible. Thank you very much! But nobody ever doubted it. On the other hand, a hundred years of symbolic logic have spoiled our taste. We want a solution of Zeno's paradox where the problem of Infinity is not carefully eliminated, but where Infinity itself takes part in the solution and provides the explanation to the riddle of motion. It is fortunate for us that Cantor's theories suggest that there is a transfinite concept of motion, and as this Cantorian idea of changing the location of an object in Space may be the mathematical basis of all future interstellar space travel it will pay off handsomely for us to have a last look at the theory of limits, and its logical shortcomings in relation to Cantor's arithmetic of Alephs.

The theory of the differential limit only means that the quantities involved are *permitted* to decrease beyond any given number. There is no end to this process. But the fact that they are permitted to approach the limit of the infinitely small does not mean that they actually reach it. On the contrary, the very fact that the process is unending demands by definition that Infinity is never reached. Otherwise this endless process would have an end which would be a contradiction in itself. Consequently, every *actual* space-interval designated by this mathematical procedure has still a *finite* extension. Newton realized that already. Permit me to translate a significant statement from the Latin text of his "Tractatus de Curvata Curvarum". It says:"… I have intended to show that it is not necessary in the method of fluxions to introduce into geometry infinitely small figures."[1] This seems to be borne out by practical experience in modern experimental physics. One of the leading physicists of our time wrote only recently: "The latest development of nuclear physics suggests that there exists a 'minimum length' below which no decrease is possible."[2]

It all boils down to the important fact that our traditional non-Cantorian system of mathematics considers Space as *having a quantized structure.* Space is made up, so to speak, of individual space-quants: tiny, discrete entities of pure extension. The limit theory only permits us to assume these space quants to be as small as we want.

If we keep that in mind, it will be possible for us to understand why we are forced to think in Zeno's paradox, that Achilles can never overtake the tortoise. In order to make comprehension easier let us again lay out our race course:



Achilles starts from A, and the tortoise begins its race at the same time from B. The finish is at Z, a point which both racers are supposed to reach at the same time; the reason being that Achilles runs twice as fast as the animal. Now, if we are required to assume that Space is quantized, then Zeno's argument will be valid and good from now to doomsday. How so? Well, let us reformulate it, and it will become evident.

Zeno argues that the two racers must occupy exactly the same number of positions during their race. The following pattern of one-one relations illustrates what Zeno means:

(I) Achilles	**A**	**B**	**C**	**D**	**E**	**F**	**G**	… Z-1
	↕	↕	↕	↕	↕	↕	↕	↕
(II) Tortoise								
	B	**C**	**D**	**E**	**F**	**G**	**H**	… Z
(III) Number of positions	↕	↕	↕	↕	↕	↕	↕	
	1	**2**	**3**	**4**	**5**	**6**	**7**	…

As one can easily see, for any number of positions, n, Achilles is invariably one position behind the animal. If our hero were to catch the tortoise, he would have to occupy *one more* position during the same period of time. This is manifestly impossible; we have to cede that to Zeno. The intervals between the tortoise and its pursuer may progressively get smaller and smaller till they reach the order of magnitude of a space-quant. But no further decrease is possible. It follows that the number of such quants or physical (not mathematical!) points between A and Z is finite. And in the case of all finite sets a subset of a series is *not* numerically equivalent with the full series (cf. part 1). Achilles must stay at least one spacequant behind the tortoise. Because for a finite series of positions Z and Z-1 are not identical.

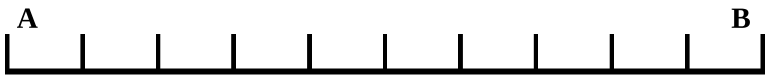
Under the circumstances there seems to be no alternative left but to assume that the "smallest" segments of the line AZ have no longer any measurable length. They must be dimensionless points. But not even the summation of an infinite sequence of such points would produce a line segment of any measurable extension. Achilles starting from point one – identified with A – and moving to the successive points 2, 3, 4, 5, 6, 7, 8... would never cover any distance. Because there is no distance between the points, and the points themselves have no spatial extension. All Achilles could do with his fastest running would be to stay in A.

Remember the story where Alice runs with the Red Queen? Let us see what Lewis Carroll had to say about it: "They went so fast that at last they seemed to skim through the air, hardly touching the ground with their feet, till suddenly, just as Alice was getting quite exhausted, they stopped, and she found herself sitting on the ground, breathless, and 'giddy … Alice looked round her in great surprise. 'Why, I do believe we've been under this tree the whole time! Everything is just as it was!' 'Of course it is,' said the Queen. 'What would you have it?' 'Well, in *our* country,' said Alice, still panting a little, 'you'd generally get somewhere else – if you ran very fast for a long time as we've been doing!' 'A slow sort of country!' said the Queen. Now, *here* you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!'" It seems Achilles and Alice run in a very fast country. Their running does not take them anywhere.

So far we have naively assumed that the number of points on our line segment AZ – the race course – is either finite or infinite. But we should not forget that the whole alternative "finite-infinite" belongs exclusively to the Cantorian number system of Aleph naught. It follows that our entire reasoning up to this point was based on the silent assumption that the structure of spatial extension – the continuum – can be adequately defined in terms of $\aleph_0$. Anybody who states that Zeno's paradox resolves into a problem of limits makes this very assumption.

The classical alternative of finite infinite was, of course, unavoidable until very recently because it was believed, that the traditional concept of the ordinary Infinite represented the upper limit for all numerical systems Cantor's discoveries have shattered this belief, and we now know – approximately seventy-five years – that the arithmetical concept of $\aleph_0$ is insufficient to give us a proper picture of the structural properties of the continuum. We are aware of the fact that the continuum can only be described with the help of the non-denumerable system of real numbers. This system is not merely infinite. It is transfinite and of the order of the cardinal number c. Under these circumstances it is evident that the general problem of motion depends in its solution on the nature of the continuum. We are bound to apply to it the arithmetic of non-denumerable transfinite c.

Naturally our next question should be: what is the difference between the arithmetic of denumerable and non-denumerable systems when used to measure distance in Space? An elementary drawing will help us again. Let



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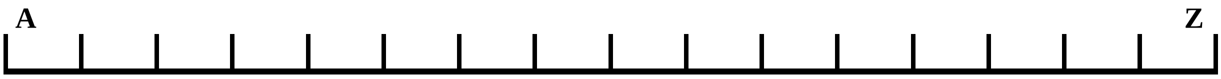


be two distances which are to be measured in terms of predetermined measuring units. And no change of the unit of measuring shall be permitted between AB and AZ. If we use denumerable numbers for our purpose, we shall find that *there is a definite relation between the number of measuring units and the length of a line segment.* In other words: the distance AB is shorter in terms of measuring units than the distance AZ. In order to indicate what we mean we might also use the ontological – existential – formulation: there are more space-quants between A and Z than there are between A and B.

This interpretation of our measuring procedure, however, is inadmissible if we define the distances AB and AZ in terms of non-denumerable numbers. We have seen in our preceding article that the order of magnitude of all real – non-denumerable – numbers between 0 and 1 is already of the transfinite cardinal order of c. The same holds for all real numbers between 1 and 2, 2 and 3, 3 and 4…, or 0 and 2, 0 and 3, 0 and 4… In fact the same quantitative relation exists generally between 0 and n, whereby n is permitted to increase without limit. In short a line a quintillionth of a millimeter long contains as many points – as designated by real numbers – as another line stretching from Earth to the last barely visible nebula in the Universe. As soon as we use the arithmetic of non-denumerable numbers we find that *there is no relation between the number of real points on a line and its length.*

As soon as we have reached this insight we are finally ready for a genuine solution of the problem of motion, as exemplified by the race between Achilles and the tortoise. It is impossible to solve Zeno's paradox in terms of a denumerable system of cardinal numbers. Achilles would never catch the animal ahead of him if there existed a rigid and invariant relation between our method of counting and the objective structure of Space. By "invariant relation" I mean a relation to the effect that the number of points we might count – no matter what technique of counting we might use – would *always* indicate the length of the measured line segment.

Zeno's thesis that Achilles must occupy as many positions as the tortoise is and remains unassailable. Equally true is that he must travel a greater distance than the animal. And if the greater distance contains more real points than the smaller one then it is impossible for him to catch up. This would be the case, indeed, if the ultimate reality of our space-time continuum could be adequately described in terms of *denumerable* numbers. Zeno discovered his paradox because he used only the denumerable numbers of the system of Aleph zero. By telling his story of Achilles and the tortoise he demonstrated the inadequacy of the classical number concept. The paradoxical situation which develops between Achilles and the animal clearly demonstrates that the problem of motion in space needs for its treatment a very different concept of number. Motion is a problem of the continuum, and therefore in its general form only treatable by an arithmetical system of transfinite magnitude. Zeno, of course, could not know this. He, therefore, drew from his absolutely correct thesis, i.e. that during the race *Achilles must occupy the same number of positions as his competitor,* the erroneous inference that in doing so he could not travel further than the tortoise. His conclusion would have been correct only if the run from A to Z actually contained more points – real numbers – than the shorter course from A to B. This, we know now, is not the case!

Therefore the solution to Zeno's problem is: *In* the *arithmetic of non-denumerable numbers Achilles occupies between A and Z no more and no less positions than the tortoise between B and Z.* Thus it is possible for Achilles to travel a longer distance than the animal although he occupies during the race the same number of positions – as Zeno correctly pointed out – as his opponent.[3]

This teaches us a fundamental lesson for all future interstellar space travel: The objective distance between two points in Space, let us say, between Earth and the Crab Nebula, can never be established by counting the absolute number of points in between. No matter how short or how long a line segment, it always has the same number of real points, and the number in question is invariably the transfinite cardinal number c. The following statement may be difficult to digest even for a willing reader, but it is true just the same: measured in the system of real numbers the distance between Earth and Crab Nebula is neither longer nor shorter than the space-interval between Earth and Moon. We naturally struggle against this revolutionary idea because we have become accustomed through thousands of years to measure distances exclusively by dint of the denumerable order of cardinal numbers. But as long as we only count denumerable numbers we cannot obtain a proper concept of either Space or Motion because both phenomena involve actual Infinity.

How little we know about the basic structure of Space in general can be deduced from the most fantastic result Georg Cantor was forced to accept when he investigated the properties of real numbers. We learned in the preceding article that

$$c^2 = c \cdot c = c \qquad (1)$$

This formula certainly looks harmless. It seems to be trivial. But it contains ontological dynamite because it means that any line – finite as well as infinite – contains as many real points as the square over it. We shall skip the demonstration of this amazing relation between a one-dimensional line and a two-dimensional plane – which, incidentally, is dome with the help of Cartesian coordinates – and proceed to the next formula

$$c^3 = c^2 \cdot c = c \qquad (2)$$

Translated into ontological terms it says no move and no less than that the number of real points of the shortest line segment is numerically equivalent to the number of all real points in an infinite three-dimensional universe.

When Cantor in 1877/78 intended to publish this almost unbelievable result the editor of the mathematical periodical refused to print his article. It took the intervention of the mathematician K. Weierstrass, who had already obtained world wide recognition for himself, to get Cantor's paper on the pages of the Journal.[4]

However, this is not all. We further know from our first article that

$$c^n = c \qquad (3)$$

This means that all real points of any n-dimensional universe – where n is a finite number – are of the same numerical order as the number of all real points of our smallest line segment. And finally we have

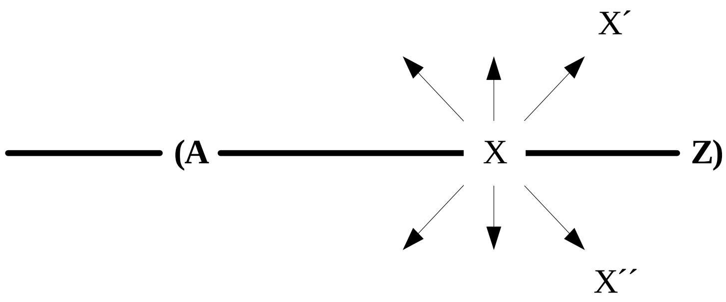
$$c \cdot \aleph_0 = c \qquad (4)$$

In plain words, the same relation would even hold if there were a universe with an infinite number of dimensions.

I am not going to discuss the far-reaching implications of the formulas (3) and (4) in this article. Formula (2) will suffice as far as our immediate problem is concerned. According to (2) it is possible to establish a one-to-one correspondence between the points of the smallest possible line segment and all spatial points of our three-dimensional universe. Our line segment has "as many" points as the entire meta-galactic Space. Let us put it differently: *The non-denumerable order of transfinite c makes spatial dimensions and distances disappear!* It should, however, be noted that the one-dimensional mapping of all space-points can only be done discontinuously. There is no continuous one-one relation between the points of a line and the points which establish the other two dimensions. That means, the transfinite system of real numbers provides us with a picture of Space which is allowed to shrink without limits. But by doing so it retains even in the form of a line segment certain characteristics which indicate that it is potentially much more than a one-dimensional sequence. The discontinuous character, of the one-one relation between a simple line segment and Space hints at the possibility that our line may at any time – provided the necessary metrical conditions exist "explode" into a three-dimensional spatial continuum.

As it is almost impossible to realize at the first reading the almost unbelievable consequences of formula (2) I shall try to outline them with a few words. First, as the shortest and the longest line segments are arbitrarily interchangeable – when measured in terms of transfinite numbers – it is metrically possible to arrive at any distance from any even point by traveling a negligible minimum length in terms of finite numbers. But the second consequence is even more fantastic. According to classical concepts of geometry one can only – traveling along a one-dimensional line – arrive at points which are located on this very line. This limitation does not exist in realms where formula (2) is valid. If a finite line segment represents transfinitely a three-dimensional continuum, then it must "contain" a transfinite number of points which are – spatially speaking – not located on the very same segment where we find them. This sounds like complete madness. But don't forget, if you had submitted the blueprints of a jet plane or a television set to Moses, Alexander the Great, or Sitting Bull these gentlemen, too, would have decided that your drawings could be nothing else but the insane products of a hopelessly diseased mind. So let us face it, formula (2) implies that any line segment "contains" points which are not located on it.

Let us assume, we travel along a line AZ, and we permit the line at a certain location X to "explode" into a three-dimensional continuum – location X, if unite, contains all the points required for the "explosion"– then, instead of arriving at point X we may arrive at, point X' or X" neither of which is located on our line of travel.



It is hardly possible to overestimate the importance of these transfinite properties of Space for theory and practice of. interstellar space-travel. We are beginning to know nowadays that our present mathematical methods do not give us an adequate picture of the dimensional Space and Time properties of our universe. They describe at best the properties of Matter *in* our world, but not the principles of extension *per se*. Therefore, they fail completely when challenged to define the basic characteristics of the four-dimensional continuum of Space and Time in which our physical existence is embedded.

Zeno's paradox makes it obvious, that our conventional ideas of distances and lengths are derived from our familiar knowledge of physical bodies. They apply to bodies, indeed, and generally to all varieties of material existence which has a quantized structure, but they do not apply to a different form of existence: the existence of the continua of Space and Time.

If Achilles could overtake the tortoise in our present world of Aleph naught – that is, if we could *think* the problem of motion by using our classical "geometrical" concept of distance, then the same concept would equally apply to interstellar distances. It is not probable that we would ever reach the stars under these circumstances. Because the idea of a journey that would take centuries even to reach our next neighbor Proxima Centauri, is absurd. And how would galactic empires – the type Isaac Asimov has described in his Foundation novels – exist, if a message from one end of our galaxy to the other side of the rim took approximately one hundred thousand years?

But interstellar travel is, theoretically speaking, an undeniable certainty because the secret of motion is that it does not happen on the basis of quantized physical conditions where distances gradually pile up to almost immeasurable orders of magnitude Everybody knows from his own practical experience that Achilles catches his animal – although the theory tells us that he cannot possibly do so. This is irrefutable proof that the quantized thinking of $\aleph_0$ does not apply to the problems of space. The continuum is of transfinite order, and here our traditional ideas about extension about distance, and about dimension become, invalid, and have to be redefined.

Thus the impending space age will force upon Man a revolution of thinking. I should like to quote several statements of John W. Campbell, Jr. which were contained in a letter (June 24, 1953) addressed to the present writer. We were discussing Cantor's theory of the transfinite Alephs, and our editor wrote [5]:

> "To date, I feel that no satisfactory correlation of Cantor's ideas with the real universe has been published.
>
> Some of the implications of Cantor's work are most disturbing to the mind orientated entirely on the quantized thinking implicit in two-valued logic, in a digital-ordered system of thinking, and in quantized physics …
>
> One of the things implicit in Cantor's work is that if any line contains Aleph-n points, then if we accept the proposition 'Things equal to the same thing are equal to each other', we must also accept that a line of any length is equal to a line of any other length! The concept 'greater than' as applied to line segments must then be re-examined.
>
> If, as Cantor's concepts imply, 'length' is a fiction derived from a limited operational method, the 'distance' between two points is purely a matter of measurement!
>
> It seems to me that there are many indications that the whole concept of geometry is a special case of something far more general, in which Cantor's concept of Aleph-null becomes simply the first-order unit.
>
> And in that system, by recognizing that distance is purely a matter of operational method … why, the stars are as near as we wish them."

Please compare these remarks with the result of our preceding article on Achilles and the Tortoise. We learned that existence and – all existence is physical existence – can only be conceived in terms of quantized thinking, be measured with digital ordered number systems, and objectively explained in quantized physics. But we also learned that all these methods fail if we want to tackle the problem of Space. The most striking indication of this failure is the existence of Cantor's formula:

$$c \cdot \aleph_0 = c \tag{4}$$

according to which the smallest line segment has "as many" real points as there are in an infinite universe with an infinite number of dimensions. This elicits, of course, the question: How small is our line segment permitted to be? There is only one *logical* answer: As small as we can measure it. And how small can we measure it? This time there is only one *physical* answer: Down to the order of magnitude of one space-quant. We are, therefore, entitled to say that one single space-quant which is, an absolute unit in terms of denumerable numbers contains as many points as any n-dimensional universe – where n is permitted to increase without limit.

Let us re-formulate this most important result from a different aspect. It is implied by (4) that our technique of measuring by gradual accumulation of length-units is valid only when applied to *physical* states of existence. It is meaningless when applied to that which "*contains*" all physical existence, i.e. to empty Space *per se.* The concept of distance is meaningful only with regard to Substance in its two manifestations as matter and energy. It does not signify anything with regard to the voidness of Space. Talking in strictly physical terms we may, therefore, say: Space *per se* does not exist. But this is by no means all there is to it. We shall learn something more by having a look at the recent history of physics.

Newton still believed in an independent "physical" existence of the absolute voidness of Space *per se.* Famous is his experiment with a rotating pail of water. Everybody knows that if a pail rotates the water will assume a concave surface. This is the effect of a centrifugal "force" engendered by the rotation, and Newton interpreted this force as the result of motion relative to absolute or empty space. The validity of his argument was first doubted by Ernst Mach. But proof that Newton must be wrong was only obtained when Michelson and

Morley performed their well-known "ether-drift" experiment and Einstein discovered its proper, interpretation. If earth moves through absolute space, then the apparent velocity of light should be greater when the observer moves towards its joint of emission, and smaller when he moves away from it. According to our classical conceptions this should be so, because in the first case one has to *add* the velocity of the observer to the speed of light, and in the second case the velocity of the observer must be *subtracted* as the light has to catch up with him. But when Michelson carried out his famous experiment no such change in the relative velocity of light was observed. No matter whether the observer moved towards the source of light or away from it the velocity of light remained constant at 186284 miles per second (in vacuum).

Classically speaking, this is perfectly absurd. Let me illustrate it with a trivial example of our everyday life. We shall assume, there are two cars on the highway, both equipped with faultlessly registering speedometers. The first car is driven at a speed of exactly 60 mph. And the second car at 62 mph. It stands to reason that the second driver will gradually overtake the first; and when he does so he will pull ahead with exactly 2 mph relative to the first car. But the negative result of the "ether-drift" experiment suggests that the second driver would pass the first car with a speed margin of exactly 62 mph. "But that is impossible!" you will say. "If the relative speed of the two cars at the moment of overtaking is 62 mph, and the first does 60, then the second car should have an intrinsic road-speed of 124 mph. It is impossible that the speedometer of second car indicates 62 mph. But in case it does, it is out of the question that the velocity of the two moving objects on our highway relative to each other is 62 mph. It is then exactly 2 mph." The argument is perfectly correct. The application of the Michelson-Morley experiment to our highway situation is nonsense. Because there *is* a highway, and both cars have two velocities – an "absolute" one with regard to the highway, and indicated by their respective speedometer readings, and a relative one with regard to each other. And their relative motion always depends on their "absolute" velocities. It can be calculated by a simple arithmetical procedure. In the case of an overtaking you subtract the smaller speed from the greater. If it is a collision, you add the two speeds to each other.

But there is "no highway in the sky!" This was Einstein's solution when he tried to reconcile the negative result of the "ether-drift" experiment with our traditional conceptions of motion. Michelson expected a positive result for his experiment because he assumed that both, the light as well as its observer, would have an absolute velocity with regard to absolute Space (ether), and in absolute Time, in addition to their relative speeds. But the experiment was negative, and Einstein concluded that there was only one explanation for its result: absolute continua have no independent physical existence.

Einstein insisted that we should distinguish between Space and spatiality, and Time and temporality. Spatiality and temporality are basic *properties* of physical events. Absolute Space and Time, however, are mere theoretical abstractions without objective reality. It stands to reason that you cannot measure abstract concepts in terms of centimeters or seconds. On the other hand, spatial and temporal properties of physical objects or processes can be measured. And this, by the way, is the scientific criterion of real existence. Nothing can be admitted as being objectively real in our Universe unless it can be measured either directly or indirectly.[6] But empty Space and eventless Time are not measurable. Their "properties" are always the same regardless of the speed or the position of the observer who is moving through them. To put it differently: It is impossible to measure distances in absolute Space or intervals of absolute Time.

As far as Time is concerned readers of science-fiction magazines are quite familiar with the relativity concept. Most of them know that, if we could travel around the whole "circumference" of the Universe in, say, a dozen years spaceship time, and we returned to Earth, billions of years would have elapsed in terrestrial time. But so far it has occurred only to a few that the distance Earth-Andromeda Nebula may be about two million light-years, measured in terms of terrestrial physics, but hardly anything, measured from a spaceship under proper space-travel conditions. This is at least theoretically possible, because distances in absolute space are nonexistent. Even more, it is meaningless to combine the idea of distance with that of empty Space because relative to absolute Space the shortest imaginable and the longest imaginable distance are numerically equivalent. This result was previously implied by Cantor's formula:

$$C + C + C + \ldots = C$$

How about space travel now? I am afraid I shall have to postpone my answer to that fascinating problem till the final article, because one important link between the different parts of our puzzle of motion is still missing.

We have established that neither Time nor Space are absolute data of Reality. We are beginning to realize that the technique of interstellar and even intergalactic space travel will probably not be hampered by the consideration of millions of years and billions of parsecs. But the physical reality of our Universe is obviously a product of *three* basic components: Space, Time and – Matter. So far we have only heard about the relativity of spatial and temporal characteristics of our Universe, but nothing about its material component. Is there also a relativity of Matter, or is material existence the absolute and irreducible core of Reality?

My third and last article intends to show that Matter is as relative as Space and Time. And it is just this reciprocal relativity of Space, Time, and Matter which will enable us to understand that interstellar and intergalactic travel is not the product of the feverish fantasy of some science-fiction writers, but a theoretically well grounded implication of modern physical science.

Footnotes:

[1] :

In the orginal text: "… volui ostendere quod in methodo fluxionum non opus sit figuras infinite parvas on geometriam inducere". Incidentally, "method of fluxions" is Newton's original name for the differential calculus.

[2] :

Cf. C. F. von Weizäcker, Zum Weltbild der Physik, 4th ed. Zürich 1949, p.145.

[3] :

A note to mathematicians: yes I know that Cantor's "positive theory of Infinite" provides a solutions to Zeno's problem only if we do not identify mathematical "existence" with construction. However, if we do – as the revolutionary school of mathematical inuitionists (Kronecker, Brouwer) insits we should – Zeno's problem is mathematically speaking still unsolved. But Kronnecker's "revolution" would banish all but the positive integers from mathematics. This seems rather larger order!

[4] :

Cf. Journal f. Math. vol.84, pp.242-252, 1878.

[5] :

Cf. Astounding, July 1954, pp. 76-88

[6] :

Cf. P. W. Bridgman, The Logic of Modern Physics. New York 1927.

TO BE CONTINUED

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ACHILLES AND THE TORTOISE

From a philosophy a science is derived; from a science, an engineering system can be deduced. From engineering come the blueprints, and the machine-shop directions. And the philosophy of present-day science, summated in Einstein's relativity, cannot lead to the interstellar cruiser. But another philosophy ...?

– part 3 of 3 –

by GOTTHARD GÜNTHER

In the pioneer days of wireless telegraphy the wife of one of the scientists engaged in developing the new invention was asked by her friends whether she could explain to them what her husband was doing. "Of course," she said, "my husband explained it to me just yesterday. Imagine a very long dog. His forelegs are in Washington when his hindlegs are still in New York. If you pinch this dog in New York, he will bark in Washington. Wireless telegraphy is exactly the same – but without dog."

Something similar could be said about interstellar space-travel. It will be space-travel all right – but without space. This seems to be a rather asinine statement. Nevertheless, we shall see later on that it expresses, in a somewhat cryptic fashion, the very secret of a possible space-travel technique.

We learned in Part 2 of "Achilles and the Tortoise" that neither Space nor Time are absolute data of our scientific experience, and that they exist only in an interdependent relation with Matter. But so far we have not yet discussed Matter. Well, what is Matter? Ancient Greek philosophy had an answer. One of its outstanding representatives (Democritus) said that Matter is the accumulation of tiny indivisible particles of an eternal and indestructible substance. All objects and phenomena of Nature can be explained by quantitative changes in the accumulations and configurations of these particles. This is the early theory of atoms. It has influenced scientific development for more than two thousand years. Recently, however, this theory has been modified, and modified to such an extent that the original idea is hardly recognizable. There are elementary particles, yes, but – there is something else as well. And, if somebody tells you that these particles are no particles at all, you won't be able to contradict him.

At first sight the situation looks rather confusing, but this is only due to the fact that we are momentarily caught in a transition period where we are passing from Democritus' idea of absolute atoms and absolute Matter to the very novel discovery that Matter is as relative as Space, Time, color, motion et cetera. Just the same, in order to find out something about the modern concept of Matter the theory of elementary particles is still a good starting point. The following table enumerates the particles which were known to physics up to the year 1953, and also shows, us some of. their properties:

Name	Charge	Mass	Lifetime	Decay Scheme	Spin
Graviton	0	0	stable	no	2
Photon	0	0	stable	no	1
Neutrino	0	0	stable	no	½
Electron	–	1	stable	no	½
Positron	+	1	stable	no	½
MesonGroup	0, –, +	210-1400	not stable	yes	mostly unknown
Proton	+	1836	stable	no	½
Neutron	0	1838.5	750 sec	yes	½
Neut.V-Part.	0	2190	not stable	yes	?
Pos. V-Part.	+	2200	not stable	yes	?

We, are rather certain that this table is not exhaustive. According to present theory there should be an "anti-proton". It is also improbable that the Meson-group is complete with its eleven members. If the present system of counting particles is retained, a final table might enumerate twenty-seven or twenty-nine particles. But this is just a hypothesis.

However, these particles do not have the properties Democritus ascribed to them. Perhaps the best way to define them is to say, that each particle represents a "localized" manifestation of a quantum field. Each quantum field, on the other hand, fills the whole of Space and Time of our present Universe. But there is a difference between these fields. Only the gravitational and the electromagnetic field are genuine long-range fields. All the others are extremely short-ranged in their observable effects. It might be said with a certain modicum of truth that the first two fields represent – partly at least – the long-range behavior of the short-range quantum fields. The gravitational and the electromagnetic field are, therefore, called the classical fields.

This abolishes completely the concept of empty Space. It looks as if Space is physically real only as an extension of the quantum fields beyond the existence of "solid" matter. This also implies that distance is a quantized property, produced by gradual accumulation of space-quants.

And what goes for Space should have its analogy in Time. There is no experimental proof yet, but it is perfectly safe to predict that there is also a smallest physical unit length of Time, an irreducible time-quant. The concept of the quant, first discovered to be a property of Matter in its fluid state as energy (Planck), is undoubtedly the general criterion of whether something is measurable or not. Space, Time and Matter – as far as they have measurable properties – are quantized. But, as far as they do not represent an accumulation of quants, they are not measurable. And what is not measurable does not exist – scientifically speaking. Or does it? Well, the equivalence between physical existence and measurability is certainly true for classical physics, but we shall obtain a very different – and most unexpected answer if we couch the available information in terms of quantum physics.

However, before I attempt to formulate the quantum-physical answer – which, incidentally, will lead us straight down the road to the problem of space-travel – permit me to recapitulate the salient features of our problem:

1) We learn from Cantor's theory of Alephs that the concept of shorter or longer distances – or interval – is arithmetically meaningless in the continuum.

2) We also know from the solution of Zeno's paradox that the phenomenon of motion is independent of the number of physical – denumerable – space-quants a moving body transverses. Achilles passes in the same time as many real points, but more space-quants than the tortoise.

3) The Michelson-Morley experiment implies that absolute Space and absolute Time are abstract relations but not physical realities.

4) The quantum field theory informs us, that the basic substratum of all physical existence are a limited number of quantum fields, each with the characteristic of extending over all of Space and through all of Time.[1]

With the help of arguments (1) and (2) the theoretical feasibility of interstellar and even intergalactic space-travel can be demonstrated. I shall show this by first analyzing the significance of these arguments. Achilles is capable of traveling, within the same time, a longer distance than the animal because by doing so he passes no more and no less real – non-denumerable – points than his competitor. Zeno assumed for the sake of simplicity that our hero travels exactly twice as far as the animal. But from what we know about the absolute equivalence of shorter and longer distances in terms of the transfinite number c [cf. argument (1)] Zeno's point would also be valid if his fast runner would travel a quintillion times as far as the slow one. The ratio of the distances is irrelevant. We therefore ask: What makes Achilles overtake the tortoise? The answer is trivial: He uses longer legs, and by doing so, he compensates for the fact that he has to pass more space-quants than his opponent.

I'm afraid we should need awfully long legs to step from here to the Andromeda Nebula. Achilles' personal method is not very practical for interstellar distances. But his example demonstrates a general principle. Distance *per se* does not mean a thing! The traversing of distance is "purely a matter of operational method" (Campbell). We have not yet reached the stars, because we use an extremely limited operational procedure for locomotion – a general procedure, by the way, which includes everything from the crawling of a toddler to the flight of a jet plane or a rocket.

Our question is: Is there a basically different type of operational method? A method more adequate to traverse cosmic distances? The answer is Yes, and it is implied by arguments (3) and (4).

If you have various forms of locomotion, some slower and some faster, but all unsatisfactory for a certain purpose, you may ask: What is their common characteristic which makes them all so inadequate? The answer in our case is simple. The toddler, the tortoise, Achilles or the jet plane, they all try to cover distances by passing space-quants. But in terms of space-quants there are always shorter and longer distances, and no matter *how* good your operational method is, there always comes a point where the number of space-quants becomes too much for your technique. And viewed from the order of magnitude of galactic distances there is hardly any difference in speed between the toddler and the jet. Both are equally outclassed in the race for the stars.

The point, therefore, is, can we envisage an operational method of locomotion which does not try to cover distances in terms of space-quants? Because as long as we do so we have to distinguish between shorter and longer distances. Arguments (3) and (4) suggest that there should be such novel method. Let us find out, therefore, what these arguments really signify concerning the possibility of interstellar travel. According to (3) absolute empty Space does not exist in a strict physical sense. Consequently, absolute distances do not exist either. What does exist is "spatiality" (Einstein) as a measurable property of matter-in-general, i.e. of all physical states of nature. So the "absolute" distance between us and, let us say, the Andromeda Nebula, which has so far made it impossible for us to visit that distant galaxy, is not absolutely real. It is only relatively real with regard to our toddler-locomotion, and it, would disappear at once if we were to discover the proper operational technique to deal with such distances.

At this point I imagine I hear some of my readers mutter: "This sounds fishy! Even apart from the problem of space-flight we know there is a stupendous remoteness between us and a distant galaxy." Surely, there is! But the point is: Do we have to interpret it in terms of spatial distance? A trip to the Andromeda Nebula performed with the velocity of light, would take approximately two million years – terrestrial time. There is nothing which could prevent our saying: There is a temporal interval between us and this galaxy. Our spacevessel rests motionless in Space, but a time interval of two million years will affect the immediate proximity of the Andromeda galaxy. Motion relative to empty Space is not observable! Motion relative to eventless Time is not observable either. It follows with inexorable logic that the two statements: I travel from here to the next, galaxy through Space or – exclusively – through Time are absolutely equivalent. For, if I travel through Space with the velocity of light, my docks will slow down to an absolute standstill.

Obviously, Space and Time are interchangeable entities. But they are interchangeable only on the basis of Matter. When I said, we travel from here to the next galaxy two million years through Time but not through Space, I measured time outside the spaceship. When I stated that we travel the same way exclusively through space and not through Time the latter was measured by the clocks of the ship. The difference *between the two sets of clocks is a material condition!* But this means there are *three* inter-connected interpretations by which the mutual relations of Earth and Andromeda Nebula may be defined. We may say:

- There is a time interval (T) between the two.
- There is a spatial distance (S) between the two.
- There is a material gradient (M) between the two.

We discovered that (T) and (S) were interchangeable, provided the material gradient (M) was represented by a constant. This constant is the velocity of light. It was assumed that our ship was traveling with that velocity.

In fact, we are quite accustomed to the interchangeability of the three cosmic components of the Universe in our daily lives! But we never think about it, and we fail, therefore, to notice the general significance of the most familiar phenomena.

In the preceding article[2] I compared the Michelson-Morley ether-drift experiment with the situation on two cars traveling on a highway, and I stated that the difference in results as to the relative velocity of moving objects was due to the fact that in the case of the two cars there existed an "absolute" constant: the highway. Now, everybody who has ever driven a car, knows that his locomotion is subject to the following laws:

$$\frac{S}{M} = T\ ; \qquad \frac{S}{T} = M\ ; \qquad T \cdot M = S$$

(where T = time interval, S = spatial distance, M = velocity = material gradient).

In the case of our highway travel (S) is always a constant, let us say, the distance between New York and Chicago. (S) is basic and cannot be changed. But (T) and (M) are variable, and to a certain degree interchangeable. We know from our experience that we can reduce the value of (T) by increasing the value of (M). In plain language: If we spend more gasoline, rubber, oil et cetera (M) we can save on time (T). But, if we want, to be economical with our engine, tires and fuel, we must pay for it in terms of longer duration. There are, of course, practical as well as theoretical limits to this experiment. Don't forget, not all traffic cops read Astounding, and are ready to accept our argument that we are only testing space-travel theories as a novel excuse for speeding.

There is, in fact, a very fundamental limitation to the operational procedure which affects the interchange between (T) and (M). This classical procedure can never be good enough to make one of the two components disappear completely, because in this case the value of the other would become infinite. But just the same, even the primitive level of terrestrial locomotion permits us to study the basic interchange-relation of (T) and (M) with (S) being an "absolute" constant.

In the case of the Michelson-Morley experiment the interchange-relation of the cosmic components is calculated on a different basis. This time the absolute constant is (M), the material factor, and (M) is represented by the most general property of Matter, the velocity of electromagnetic waves. Consequently we are entitled to expect in this case an interchange-relation between (T) and (S). This is indeed confirmed by the transformation equations of H. A. Lorentz:

$$t' = \frac{t - \frac{v}{c^2} \cdot x}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad x' = \frac{x - v \cdot t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

If we have two systems K and K' being in rectilinear uniform motion relative to each other, then the following relation exists between temporal intervals (T) and spatial distances (S) according to Einstein's interpretation of the formulas /2/: The faster K moves relative to K', the longer will be the time interval transferred from one to the other system, and the shorter will be the spatial distance in the same transfer. At the same ratio as time grows, Space tends to disappear. And vice versa. The basis of this Space-Time interchange-relation is mass or Matter, because at the velocity of light the increase in mass for any moving body becomes infinite.

So far we have obtained two very important results! If we use spatial distance (S) as a basis, we obtain an interchange-relation

$$T \leftrightarrow M \tag{1}$$

If matter (M) is considered to be the constant we find that

$$S \leftrightarrow T \tag{2}$$

holds. Having arrived at this point we can no more evade the question: Is there a third interchange-relation where (T) is the basic constant? My answer is a very positive Yes. There must be a third exchange-relation

$$M \leftrightarrow S \tag{3}$$

because the mutual interdependence of (S), (M) and (T) is such that the first two interchange relations could never exist unless a third between (M) and (S) would balance them. (We shall later demonstrate this in terms of symbolic logic.)

Interchange-relation (3) is the most interesting for us. We know (1) only to the limited extent of our daily experience in terrestrial travel. (2) Is at the moment a purely metric problem between the two continua Space and Time. But (3) represents the very core of modern quantum-mechanical physics. It is becoming more and more difficult nowadays to draw a distinctive line between what is purely "spatial" and what is "material" in any physical datum. The borderline between Space and Matter shows a sharp demarcation in classical physics only. It tends to disappear under the "boundary" conditions of microphysics as well as of astrophysics. It is only in the intermediate field of "terrestrial" macrophysics that we *seem* to know with some certainty what the difference between empty Space and solid Matter amounts to. In popular language: It is the difference between something and nothing. Matter is the sum total of "all things", and empty Space is the total absence of things. Every child can understand this. But not too long ago a most important formula was discovered: (Einstein)

$$E = m \cdot c^2$$

(where E = the energy of a body at rest, m = its mass and c = velocity of light). Since then it has become general knowledge that "solid matter" (mass) may be transformed into energy (atom bomb), and that it is at least theoretically possible to re-transform radiant energy (light) into Matter. Does our distinction between Some*thing* and *No*thing still hold? Is energy a thing? We are told that Matter is an electromagnetic phenomenon. But the electromagnetic field extends through all of Space. Space itself appears to be a field-phenomenon. That means, microphysically speaking, it becomes more and more impossible to draw a sharp line of demarcation between a thing and the space that surrounds it.

An analogous intimate relation between Space and Matter exists in macrocosmic physics. A preliminary remark: Imagine you have a box partly filled with marbles. You take one after the other of these marbles out of the container till there is nothing left *inside*. Nobody will doubt that there is one thing left. That is the empty box. Its capacity of being a container has not been affected by the removal of the contents. This idea of the relation between Space and its contents is that of classical macrophysics. But now let us proceed to macrocosmic physics.

Imagine yourself to be an entity with divine power, and located outside the Universe. You reach into the Universe and remove from it one galaxy and one nebula after the other, and you continue to do so till nothing material, not even the smallest meteorite or the most tenuous wisp of cosmic dust is left in the Universe. According to common sense two objects should then be left: the empty Space waiting to be filled again with things, and empty Time waiting for the event of a new creation. Common sense tells us further that the dimensions of our Universe should not be affected by the removal of Matter and of events. But common sense has failed us once before when we dealt with Cantor's theory of transfinite number, and it will fail us again in macrocosmic physics.

We shall throw only a fleeting glance at what happens to Time in a Universe deprived of all Matter. In our present Universe Time has two directions. It stretches toward the past as well as toward the future. But in an empty Universe Time would have only one direction – toward the future. A past would not exist. The possibility of "passing" Time demands the presence of Matter.

But what about Space? Some time ago the English physicist Sir Arthur Eddington tried to define the interdependency of Space and Matter in certain equations – in a somewhat similar way as the Maxwell-Hertz field theory connects the electromagnetic fields with charges or poles. Eddington's equations permitted two interpretations which became known as, the Einstein- and the de Sitter-universe. Einstein's universe is "static"; de Sitter's is in constant expansion – and contains no Matter! Consider what that means! A space that "contains no matter" but is in constant expansion *is* Matter in some state of radiant energy. It was later discovered (Friedmann, Lemaitre, Robertson) that Eddington's equations allow of a series of solutions which define a connection between the extremes of the Einstein- and the de Sitter-universe.

The gist of the theory can be described as follows: If you put some Matter into the de Sitter-universe, then its gravitational energy will start to counteract the expansion. The expanding world will start to slow up. If you add more and more material, you will finally arrive at a point where expansional and gravitational forces balance each other. This is the static world of Einstein. But if you add even more Matter, then gravitation becomes stronger than expansion, and the Universe will start to shrink.

It seems as if Matter has a "contracting" effect on Space, and a corresponding influence on the structure of Time. As far as Space is concerned one of the ways to describe its relation to Matter is represented by the formula:

$$\frac{K \cdot M}{c^2} = \frac{1}{2} \cdot \pi \cdot R \qquad (5)$$

(K = gravitational constant; M = mass of Universe; c = velocity of light; π = ratio of circumference to diameter; R = radius of curvature.)

As (5) is at present a highly speculative formula I shall not deal in detail with it. But, even if it is only a rough approximation of the truth, it clearly implies that distance is a field-effect. By increasing or decreasing this effect spatial distances can be shortened or made longer. This seems probable anyhow. But distances in our universe are so enormous that even a considerable reduction of spatial dimensions would not help us much. What would be the use of pulling a galaxy which is about one billion light-years away into our immediate "neighborhood" of fifty million parsecs. Moreover, we would need the energy of millions or billions of galaxies to do it. The very idea is the height of absurdity.

Fortunately, our basic interchange-relations (1), (2) and (3) suggest something else. Some way back in these articles I stated that Space and Time are continua; but Matter has a discontinuous structure it is quantized. Then a very confusing thing happened: We found out that Space was quantized too, when we traveled through it. Zeno's paradox was produced by the conflict between denumerable (quantized) and nondenumerable numbers. How did that happen? The answer is simple: All classical forms of motion demand a partial "materialization" of Space. We travel through Space by means of a highway, for instance. And a highway, representing distance, as a functional part of an act of locomotion, is materialized space. And distance in form of a highway is, of course, quantized. For the airplane our atmosphere plays the corresponding part. And even the principle of rocket propulsion – which may suffice for the short hop to, the Moon – is still based on the idea of quantized motion. It is only a little more sophisticated – you take your "highway" along with you. This time it is your rocket tubes.

But this convertibility of the spatial continuum into quantized material existence is only possible if there is a reversed process by which quantized material distance can be converted into a non-quantized state of continuity. We have at the moment not yet the slightest idea how this can be managed. Nevertheless we know two things with absolute certainty. We know that there can be no doubt that this inverse convertibility from quantized into non-quantized existence does exist. Because our three interchange-relations,

$$T \leftrightarrow M$$
$$S \leftrightarrow T$$
$$M \leftrightarrow S$$

would not be possible if only: Space could turn up as a quantized form of Matter, but Matter never as a spatial continuum Matter *has* spatial extension, in quantized form nobody ever doubted it. Our whole technology is based on that knowledge. But, if the inverse relation between Matter and Space did *not* exist we would have to assume that Matter – and with it

Time – would constitute absolute data of nature. No modern scientist is ready to make this concession.

The other thing we know with equal certainty is that in a technology no longer based on the discreteness of Matter but on the continuity of Space and Time distance goes by the boards. There is no longer any question of shorter or longer distances. They disappear completely. A line segment one billionth part of a millimeter long is numerically equivalent to a line segment of any trans-cosmic length.

Only a "little" question is now left. How can we develop a technique of locomotion which does no more use Space in its quantized material aspect, but which utilizes Matter in its, non-quantized spatial version? We are fortunate in so far as we are not totally ignorant in this respect. We have at least some negative knowledge – we are aware of the data that are missing. First: We shall have to discover the law which describes the physical interaction between the electromagnetic and the gravitational field. We need, further, detailed information about the cosmic "glue" that holds the atomic nucleus together (Meson theory).

One scientist (H. Bethe) has recently expressed the opinion that we need more powerful mathematical tools to tackle the problem of the nucleus. It may well be that Cantor's theory of non-denumerable sets will finally provide the answer. It should not be forgotten that Cantor only discovered the *existence* of transfinite numbers. How to use them in physical science is still a mystery to us. In the preceding article I could only show that non-denumerable sets do apply to the problem of motion in Space, and that their application demonstrates that distance is a property of quantized Matter but not of a continuum like Space or Time.

Thus these articles on the problem of interstellar and intergalactic space-flight would have to conclude on a very unsatisfactory negative note, if it were not for a recent discovery in the field of applied symbolic logic. We are now well acquainted with the fact that symbolic logic can be used in analyzing electrical circuits and power patterns. I shall use this technique in order to demonstrate the basic power pattern all interstellar spaceships will have to apply. I know, of course, nothing about the details which will depend entirely on as yet unknown operational procedures of electromagnetic, graviton, and messianic character, but I know by logical analysis that they will have to conform to the following structural pattern.

"M," "S" and "T" shall again be the symbols representing our three cosmic components of the Universe. Their interchange-relation shall be represented by ↔', ↔'' and ↔''' respectively. We then obtain three elementary principles of interchange:

↔'		↔''		↔'''	
M	S	S	T	T	M
S	M	T	S	M	T

Table I

We know, on the other hand, that any physical event involves all three cosmic components; because, it is physical, and it happens in Space as well as in Time. That means: No isolated interchange relation of ↔, ↔', or ↔'' ever happens in the Universe. We, therefore, introduce two operational procedures: $\overrightarrow{op}$ and $\overleftarrow{op}$ with the intent to combine all three interchange-relations. The choice of these two procedures is by no means arbitrary. In fact they are the only possible operations, if we want to combine all three interchange-relations.

We shall define $\overrightarrow{op}$ as the combination of the upper line of all three relations of ↔; that means in this case: of M to S, S to T and T to M.

The second operation, $\overleftarrow{op}$, will accordingly combine the lower line of ↔ relations in Table I.[3]

In order to establish our operational procedures $\overrightarrow{op}$ and $\overleftarrow{op}$ we arrange the cosmic components M, S and T in all possible combinations in two horizontal lines X and Y (cf. Table II). We then look for the values of $\overrightarrow{op}$ in the upper lines of our ↔', ↔'' and ↔''' tables, and we always take the *second* value. Thus, if our two horizontal lines contain the combination M|S or S|M we choose the value S from ↔' table. If the combination is S|T or T|S we choose again the second value of the upper line, this time of ↔''. Accordingly our value must be T, and so on. For $\overleftarrow{op}$ we select the *second* values of the lower line, this time of the ↔ tables. We thus obtain as definitions of the operations $\overrightarrow{op}$ and $\overleftarrow{op}$ the following table:

X	M	M	M	S	S	S	T	T	T
Y	M	S	T	M	S	T	M	S	T
X $\overrightarrow{op}$ Y	M	S	M	S	S	T	M	T	T
X $\overleftarrow{op}$ Y	M	M	T	M	S	S	T	S	T

Table II

This arrangement may seem redundant, to the uninitiated. But it is not. As a basis for further calculations we need all logically possible combinations, even such seemingly redundant ones as between M and M, S and S, T and T.

One thing is absolutely certain – no matter how interstellar and intergalactic space-flight achieved, and what detailed technical arrangement may be used – all issues space-ships capable of traveling cosmic distances will have a switchboard based on the two operational patterns of $\overrightarrow{op}$ and $\overleftarrow{op}$. The interesting point, of course, is what could be done with it. Well, many things. It is impossible to predict which of the individual technical procedure that fall into the pattern of $\overrightarrow{op}$ and $\overleftarrow{op}$ will finally be used. This depends entirely on what future discoveries will be made in nuclear and astrophysics. But we might as well – for the sake of the practical demonstration of our "switch board" – assume that the description of interstellar flight, given by one of our outstanding science fiction authors is approximately correct. Permit me to quote this description from one of Isaac Asimov's novels, where one of the officers of a spaceship explains the principles of interstellar flight to the passengers[4]:

> "Ladies and gentlemen! We are ready for our first Jump … The Jump is exactly what the name implies. In the fabric of space-time itself, it is impossible to travel faster than the speed of light … Therefore one leaves the space-time fabric to enter the little-known realm of hyperspace, where time and distance have no meaning. It is like traveling across a narrow isthmus to pass from one ocean to another, rather than remaining at sea and circling a continent to accomplish the same distance.
>
> … Great amounts of energy are required, of course, to enter this 'space within space' as some call it, and a. great deal of ingenious calculation must be made to insure re-entry into ordinary space-time at the proper point. The result of the expenditure of this energy and intelligence is that immense distances can be transversed in zero time. It is only the Jump which makes interstellar travel possible."

Asimov like many other science fiction writers sees quite clearly that the possibility of space-travel depends on the elimination of spatial distance. We know now that distance must be interpreted as an accumulation of space-quants, and we travel distances by passing space-quants. I remarked in Part 2 of this series: No matter *how* good our travel methods might become, there is always a critical point where the accumulated number of space-quants becomes too much for our operational methods. The question, therefore, is: Will it be possible to devise a technique of locomotion where we "jump," as Asimov says. In other words: Where we do not pass space-quants when we cover distance.

We know that what is quantized is the M-factor in the Universe. Neither Time nor Space, being genuine continua, are quantized *per se*. It is only their matter-aspect that gives them a quantized structure. It all boils down to the problem: Can we get rid of the M-factor in our operations $\overrightarrow{op}$ and $\overleftarrow{op}$. Our highway experience has already taught us that M and T might be traded against each other – at least to a limited degree. The same applies to S and T. But if there is general convertibility of all three cosmic components into each other, it should indeed be possible to eliminate one of the components operationally. Asimov's description suggests that the M-component should disappear, because it is the matter-factor that produces distance.

There is a very simple procedure by which the quantized M-factor can be eliminated. For $X\overrightarrow{op}Y$ it is expressed by the formula[5]:

$$\overrightarrow{op} \leftrightarrow' ((X\overrightarrow{op}Y)_{res}, (X\overrightarrow{op}Y)_{res}) \qquad (6)$$

By using (6) we transform our original sequence for $X\overrightarrow{op}Y$ from Table II:

$$(X\overrightarrow{op}Y)_{res}: \qquad \mathbf{M\ S\ M\ S\ S\ T\ M\ T\ T} \qquad (6a)$$

into the M-free sequence

$$\mathbf{S\ S\ S\ S\ S\ T\ S\ T\ T} \qquad (6b)$$

This new series which contains only S and T represents the symbolic meaning of (6): However, we are more interested in its practical significance for interstellar travel. Seen from this angle, (6) describes the basic logical pattern of a technical operation which has ceased to use quantized data. We remember that quantized data are always represented by M. But (6) does no longer contain an M-factor. (Incidentally, (6) does *not* mean that Matter *per se* has been eliminated, but only its basic feature of quantization.) It follows that the mathematical theory of (6) is not based on denumerable arithmetics. Instead, it uses transfinite Cantorian numbers. And we remember that the concept of distance disappears entirely in transfinite arithmetics. In other words, (6) describes the energy pattern of a technical operation which eliminates distance.

It can be said with absolute certainty that (6), its corresponding formula for the inverse operation $X\overleftarrow{op}Y$

$$\overleftarrow{op} \leftrightarrow''' ((X\overleftarrow{op}Y)_{res}, (X\overleftarrow{op}Y)_{res}) \qquad (7)$$

or any other expression of the types (6) and (7) will be the basic formulas of all future interstellar or even intergalactic space-travel – provided, of course, that our general assumption of the universal convertibility of Space, Time and Matter is correct. If it isn't, we might as well say good-by to all our dreams of space-travel outside our solar system. The distances are too great. The idea of interstellar voyages where space vessels travel for centuries, and where only the great-grand children of the original travelers arrive, is perfectly absurd. Interstellar and intergalactic space-travel will become a reasonable proposition only if we develop a technique which makes distance go by the boards – entirely. This, however, is absolutely predicated on the assumption of the primordial interchangeability of the three cosmic components of the Universe, and the additional assumption that one may be substituted by the other.

There are two components which are continua, and one which is quantized. Length of distance or length of interval means nothing in the continuum. It is significant only for the third (quantized) component. If we are capable of developing a technique which eliminates the quantized component by substituting non-quantized features, our problem how to cross the voids of Space is solved. Our formulas (6) and (7) demonstrate that such a technique is *logically* possible. What (6) and (7) imply *practically* can be put down in very simple words. Everything that exists has *three* components. The three *natural* components of the Universe are M, S and T. But if Man introduces an *artificial* state of existence he likewise needs three components only. But, being artificial, this state of existence is based on an operation "op" – which, in its turn, *acts as a basic component*.

Thus – introducing P for $\overrightarrow{op}$ and $\overleftarrow{op}$ – we have now four equally fundamental parameters:

P-M-S-T

This adds up to a redundancy, as far as human technique is concerned. We may omit one of the four components – or rather its properties. If we get rid of P, we have the Universe just as it is without Man's action. Of, course, P is still there as far as the objective world is concerned but it is *distributed* over MST. Process (P) appears in this case only as a natural event like the falling of rain, lightning, hunger, thirst, et cetera. But, if we introduce an independent parameter P as a fourth degree of freedom for action, we add human creative procedure to the natural events – as a means of producing something which has not existed before. In our special case – space-flight But the addition of P permits us to eliminate the properties of one of the other components by substitution. Our formulas (6) and (7) did this for the pattern: P-S-T

This is possible because all human technique and action is *two-valued*. In order to do something we require operational decisions and two alternative poles to decide between. Nothing else! Therefore, the pattern PMST is always redundant – as far as any individual technical procedure is concerned. M (or the properties of M) can be substituted as well as S,T or P.

However, this redundancy concerns only any practical, i.e., limited, action or procedure, and *not* the general rational pattern on which our procedure is based. That is the reason why we cannot be satisfied that science, as it is known today, recognizes only three parameters of the Real:

the objective parameter:	Matter (M);
the dimensional parameter:	Space (S);
the relational parameter:	Time (T).

It is necessary to add a fourth cosmical component:

the operational parameter: Process (P).

As long as we refuse to accept P as an independent parameter we can only build a technique based on the natural laws inherent in the relations between M, S and T. In this case we will never reach the stars. The velocity of light as upper limit for the propagation of physical events in space is such a natural law. And mind you, it will *never* be invalidated in a MST-universe. On the other hand, if we add a fourth parameter we gain a technical dimension in which the laws of a three-parameter-world are not abolished but capable of – modulation.

I started my "Achilles And The Tortoise" series with an analysis of the problem of motion, as first conceived by Zeno. Permit me therefore, to conclude this final installment with a confrontation between the theory of motion in a three- and four parameter-universe. According to classical principles the phenomenon in question is based on the formulas. S/M=T, S/T=M, MT=S. These are expressions of basic laws in a three-parameter-universe (MST). Of course, P is also present in these three formulas, but as I remarked before, it is distributed over the other parameters, and, therefore, has no quotable value as an independent component. This situation is unavoidable, in classical science.

This science can never provide us with an adequate theory of interstellar space-flight because the *distributed* P cannot absorb the quantized character of M. And nothing can be done about it because, in a MST-universe P can never be taken out of its distribution. Why not? Well, according to our traditional ideas Matter is Some*thing* and empty Space and eventless Time are Nothing. But what is Process (P)? A very embarrassing question! It certainly is nothing! But is it a thing? No, we cannot admit that either. The science we know uses a strictly two-valued logic. In other words, there can be no basic Third between Some*thing* and *No*thing. In this system there is no room for an independent P. Consequently the only method to find an asylum for the fourth parameter was its distribution over the other components.

Now we have finally discovered the original source of Zeno's trouble: He could not define motion because it was neither a thing nor a no-thing. Motion is an event or process. To put it differently: Zeno's and our analysis of motion showed that this phenomenon could not be defined in terms of M, S and T. It demonstrated properties beyond the three-parameter reality of our traditional idea of Nature. We, therefore, introduced a fourth parameter P. This was done arithmetically by using Cantor's transfinite number c for the solution of Zeno's Paradox. This provides, us with a new interpretation of Cantorian transfinite sets. Cantor's Alephs are numbers which do not apply to a three-parameter-universe. They represent the arithmetical order of the fourth, parameter. This gives us – this is at least the hope of the present writer – a "satisfactory correlation of Cantor's ideas with the real universe" (Campbell).

The recognition of Process (P) as a new parameter beside Matter, Space and Time is equivalent to introducing in experiential terms that so-much-talked-about "fourth-dimension", *through which the first three can be rotated at will*. This method permits the substitution of the properties of one parameter by those of the three others. The development of a physical science which satisfies these conditions is now only a question of time. And when this time arrives – "Why, the stars are as near as we wish them."

Footnotes:

[1] :

However it should not be forgotten that this concept of Time refers only to the temporal duration of our *present* Universe, where T is now of the approximate order of magnitude of 3·1017 sec.

[2] :

In these equations v represents the velocity of K relative to K', c is the constant of light, x indicates the movement along the x-axis, and t represents time.

[3] :

A combination of a ↔ relation of the upper line with one of the lower line is not possible because it would involve our operational procedures into contradictions. An example may demonstrate this. If we combine M to S with T to S in one operation we lose the distinction between M and T, and the ↔''' relation becomes nonexistent. Because, if M is changing into S it cannot also change into T within the same operational procedure. The same holds for T.

[4] :

cf. Isaac Asimov, The Stars, Like Dust. N.Y., 1951 (Doubleday), pp.39/40.

[5] :

The transformation effected in (6) is quite elementary. It is done in two steps (note_evgo: the notation (6) was slightly changed compared to the original text):
First, starting with M for X, Y (Table II) the result for $(M \overrightarrow{op} M)$ is M which is the first value of $(X \overrightarrow{op} Y)_{res}$ in (6a). The interchange relation ↔' yields the combination $M|S$.
Second the procedure $\overrightarrow{op}$ which was originally operative only between X and Y is applied on the resulting combination $M|S$ which according to Table II gives S the first value in (6b).

THE END

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