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Title

Complementary Calculi: Distinction and Differentiation

A further step to a graphematic turn in the construction and understanding of calculi

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Abstract

The paper "Diamond Calculus of Formation of Forms.–A calculus of dynamic complexions of distinctions as an interplay of worlds and distinctions” was mainly based on a deconstruction of the conditions of the calculus of indication, i.e. the assumption of a “world” and “distinctions” in it. The present paper “Calculi of Indication and Differentiation” opts for a graphematic turn in the understanding of calculi in general. This turn is exemplified with the George Spencer-Brown’s Calculus of Indication and the still to be discovered complementary Mersenne calculus of differentiations. The proposed study is restricted, mainly, to the mono-contextural case of dissemination of calculi.

First steps toward a graphematics had been presented with “Interplay of Elementary Graphematic Calculi. Graphematic Fourfoldness of semiotics, Indication, Differentiation and Kenogrammatics". Graphematic calculi are not primarily related to a world or many worlds, like the CI and its diamondization. Graphematic calculi are studying the rules of the economy of kenomic inscriptions. Graphematics was invented in the early 1970s as an interpretation of Gotthard Gunther’s keno- and morphogrammatics, inspired by Jaques Derrida’s grammatology and graphematics. A new approach to a formalization of the calculus of indication and the calculus of differentiation too, is proposed.

Spencer-Brown’s calculus of indication has been extensively used to interpret human behavior in general. The proposed new complementary calculus to the indicational calculus, the Mersenne calculus, might not be applicable to human beings, but there is a great chance that it will be a success for the interaction and study of non-human beings, such as robots, aliens, and Others. See also: 3_15

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Categories of the RK-Archive

Complementary Calculi: Distinction and Differentiation

A further step to a graphematic turn in the construction and understanding of calculi

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Abstract

The paper *"Diamond Calculus of Formation of Forms. A calculus of dynamic complexions of distinctions as an interplay of worlds and distinctions"* was mainly based on a deconstruction of the conditions of the calculus of indication, i.e. the assumption of a "*world*" and "*distinctions*" in it. The present paper "*Calculi of Indication and Differentiation*" opts for a graphematic turn in the understanding of calculi in general. This turn is exemplified with the George Spencer-Brown's Calculus of Indication and the still to be discovered complementary Mersenne calculus of differentiations. The proposed study is restricted, mainly, to the mono-contextural case of dissemination of calculi.

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A new approach to a formalization of the calculus of indication and the calculus of differentiation too, is proposed.

Spencer-Brown's calculus of indication has been extensively used to interpret human behavior in general. The proposed new complementary calculus to the indicational calculus, the Mersenne calculus, might not be applicable to human beings, but there is a great chance that it will be a success for the interaction and study of non-human beings, such as robots, aliens, and Others.

SHORT VERSION (work in progress v.0.8.5)

1. Indication and differentiation in graphematics

1.1. Bracket Grammars

Moshe Klein has given a simple introduction to George Spencer-Brown's calculus of indication (CI) as a special case of a bracket grammar. A context-free language with the grammar: S --> SS|(S)| λ is generating the proper paranthesis for formal languages.

What was an act of a genius becomes an *adhoc* decision to restrict the grammar of bracket production. Set the restriction of bracket rules to: (()) () = () (()) and you get the basic foundation of the famous CI as introduced by George Spencer-Brown. Nobody insists that this is an appropriate approach but it seems that it takes its legitimacy from the formal correctness of the elaboration.

The law of complementarity

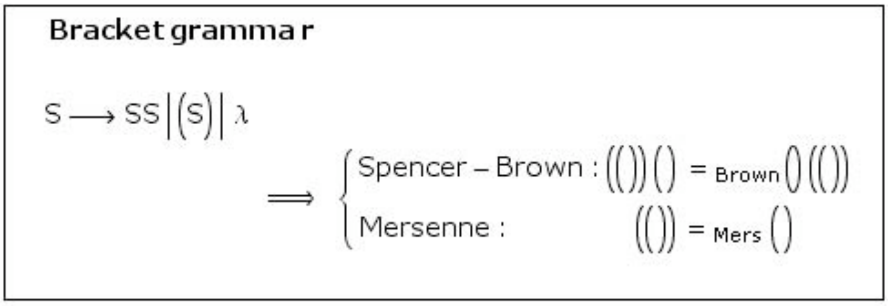
"There is no stronger mathematical law than the law of complementarity*. A*

thing is defined by its complement, i.e. by what it is not. And its complement is defined by its uncomplement, i.e. by the thing itself, but this time thought of differently, as having got outside of itself to view itself as an object, i.e.`objectively', and then gone back into itself to see itself as the subject of its object, i.e.`subjectively' again.” (George Spencer-Brown, Preface to the fifth English edition of LoF)
http://www.hyperkommunikation.ch/literatur/spencer-brown_form.htm

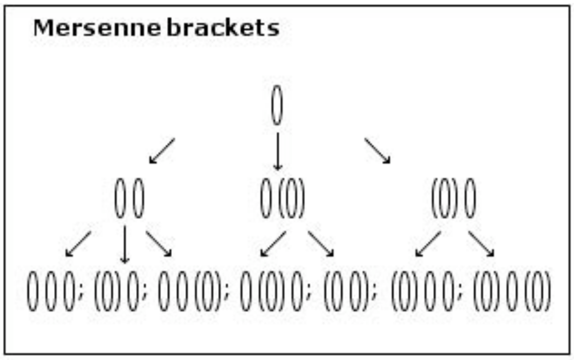
Now, with the same decisionism, albeit not pre-thought by a genius, I opt for an alternative restriction, (( )) = ( ). This decision is delivering the base system for a Mersenne calculus, interpreted as a calculus of differentiation, CD.

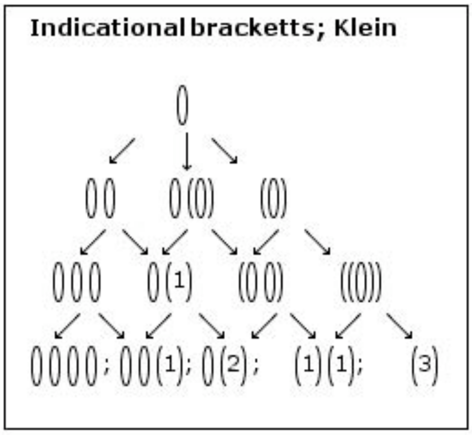
I stipulate that both calculi, the CI and the CD, are complementary. And both calculi have additionally their own internal *duality*, delivering the dual calculi, i.e. the dual-CI and the dual-CD.

It will be shown that, despite of its non-motivated *adhocism*, both calculi are well founded in graphematical systems, and are to be seen as *interpretations* of independent complementary graphematical calculi. In fact, they belong, with the identity system for semiotics to the only two non-kenogrammatic graphematical systems of the general architectonics of graphematics.

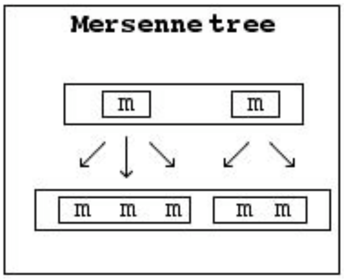


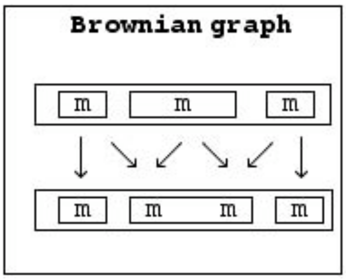
Production systems





Production schemes





Rule systems

Numeric indicational rules

R1: $\Longrightarrow \{1^1\}$

R2: $\{1^1\} \Longrightarrow \{1^2\} | \{1^1 2^1\} | \{2^2\}$

R3.1: $\{1^n\} \Longrightarrow \{1^{n+1}\} | \{1^n 2^1\}$

R3.2: $\{2^n\} \Longrightarrow \{2^{n+1}\} | \{1^1 2^n\}$

R3.3: $\{1^n 2^n\} \Longrightarrow \{1^{n+1} 2^n\} | \{1^n 2^{n+1}\}$

Numeric Mersenne rules

R1: $\Longrightarrow \{1^1\}$

R2.1: $\{1^n\} \Longrightarrow \{1^{n+1}\} | \{2^n 1^1\} | \{1^n 2^1\}$

R2.2: $\{1^n 2^n\} \Longrightarrow \{1^n 2^n 1^1\} | \{1^n 2^{n+1}\}$

R2.3: $\{2^n 1^n\} \Longrightarrow \{2^n 1^{n+1}\} | \{2^n 1^n 2^1\}$

2. Brownian and Mersennian Calculi

2.1. Towards Spencer-Brownian and Mersennian calculi

2.1.1. Primary lessons out of the bracket systems

LoF		**Mersenne**
distinction	--	differentation (separation)
indication	--	identification

Indication:

J1 : $\{\}\{\} = \{\}$

J2 : $\{\{\}\} = \emptyset$

Mersenne:

M1 : $[\,][\,] = \emptyset$

M2 : $[[\,]] = [\,]$.

2.1.2. Brownian calculus

J1 $\urcorner\urcorner = \urcorner$

A distinction of 2 distinctions is a distinction.

J2 ⏋⏐ = ∅

A distinction of a distinction is no distinction.

In Spencer-Browns wording:

"A1. The value of a call made again is the value of the call. *Calling*

A2. The value of a crossing made again is not the value of the crossing. *Crossing*"

"In his Laws of Form *(hereinafter LoF), in print since 1969, George Spencer-Brown proposed a minimalist formal system, called the primary arithmetic, arising from the primitive mental act of making a distinction. He reached the next rung on the ladder of abstraction by letting letters denote, indifferently, a distinction or its absence, resulting in the primary algebra. The primary arithmetic and algebra featured a single primitive symbol '⏋' in LoF."* (Meguire)

2.1.3. Mersenne calculus

M1 : ⏌ ⏌ = ∅

A differentation between 2 differentiations is an absence of differentiation.

M2 : ⏌⏌ = ⏌

A differentiation of a differentiation is a differentiation.

Paraphrase

M1. The value of a call made again is not the value of the call. *Calling*

M2. The value of a crossing made again is the value of the crossing. *Crossing"*

"In his Laws of Differentiation (hereinafter LoD), in print since 2011, Rudolf Kaehr proposed a minimalist complementary formal system to the LoF, called the primary complementary arithmetic, pca, arising from the primitive scriptural *act of perceiving a* differentiation*. He reached the next rung on the ladder of* graphematic *abstraction, the Mersenne calculus, MC, by letting characters* inscribe*, differently, a differentiation or its absence, resulting in the primary complementary algebra, CD. The primary complementary arithmetic and algebra featured a single primitive symbol '⏌' in LoD."* (Kaehr)

A morphic turn for LoD and LoF

A Morphic Turn for LoD

"In his Morphic Laws of Differentiation *(hereinafter MorphLoD), in print since 2012, Rudolf Kaehr proposed a minimalist complementary formal system to the LoF, called the primary complementary morphic* arithmetic*, morph-pca, arising from the primitive* scriptural *act of perceiving* patterns *of* differentiations*.*

He reached the next rung on the ladder of graphematic *abstraction considering the morphic structure of patterns of differentiations and absences of differentiations, the morphic Mersenne calculus, morphMC, by letting* patterns *of characters* inscribe*, differently, a differentiation or its absence, resulting in the primary complementary morphic* algebra of differentiation*, morphCD.*

The groups of differentiations, called situations*, are defined by the Mersenne distribution of elementary differentiations with the combinatorial formula:* 2^n-1*. Such groups are embedded into differential contexts.*

The primary complementary arithmetic *features two primitive symbols '⏌' and "∅" in LoD.*
The pattern or morphic approach is motivated by the fact of the morphogrammatics of the MorphLoD as being a part of the graphematic writing systems.

While the primary complementary morphic algebra *features a* group *of primitive constellations of symbols, '⏌' and '∅', in morphLoD."* (Kaehr)

A Morphic Turn for LoF

The groups of distinctions, called constellations*, are defined by the Brownian distribution of elementary distinctions with the combinatorial formula:*

$$\text{Brown}_{(n,\, m)} = \binom{n + m - 1}{n}.$$

Accepting the pattern-oriented approach as primordial, it is just a natural step to the development of an element-oriented approach of the classical CI and the proposed CD in its first presentation as a an element-oriented calculus too.

systems	element	pattern	mixed
indication	CI	morphCI	□
differentiation	CD	morphCD	□
□	□	□	□

Some further introductions in German at:
http://www.thinkartlab.com/Memristics/Komplementaritaet/Komplementarität%20in%20der%20Graphematik.pdf

2.1.4. How to draw a difference between distinction and differentiation?

Graphematics of Mersenne and Spencer – Brownian calculi

Mersenne : model(M) = {(aa), (ab), (ba)}

Spencer – Brown : model(B) = {(aa), (ab), (bb)}.

Complementary graphematics

model(M) : (aa) (ab) (ba)

↕ compl ↙ ↘ ↖ ↗

model(B) : (aa) (bb) (ab)

Comparison models

$M = (m, M_1, M_2)$

$B = (b, J_1, J_2)$

and $m = \{m_1, m_2\}$, $b = \{b_1, b_2\}$.

$m, b = \{a, b\}$

Theorem

M sim J iff M_1 sim J_2, M_2 sim J_1.

Proof

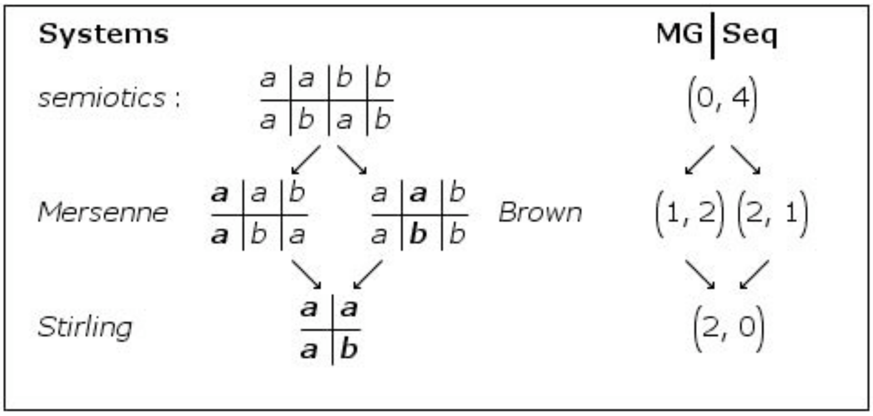
Axioms J and M are taken as operators. Sets of elements are *b* and *m*.

$J_1(m_2) = b_1$, $J_2(m_1) = b_2$

$M_1(b_2) = m_1$, $M_2(b_1) = m_2$

Morphograms and sequences

A further difference between the Mersenne and the Brown calculus is naturally given by the difference of morphograms, MGs, and sequences, Seqs, or monomorphies and constellations of the configuration for m=n=2.



Second – level partitions and the reflectional property $Ord_{refl}(m)$

Another differentiation is achieved with the question of *second-level partitions* (Moshe Klein) for the graphematics systems of the different calculi. Because of the minimal complexity of the introduced CI and CD with m, n=2, an application of the question of second-level partitions is not delivering any useful results. Nevertheless, a reflection on the notion of partition, i.e. a partition of partition, is an interesting feature for any calculus. With an extension of the complexity even to just m=3, the covered features are appearing clearly.

It should be reflected that a *first*-level and a *second*-level operation of addition based on first-order and second-order distinctions has to be differentiated. In a term like "{1 +{{1+1}}", that is representing the bracket-notation for the CI, there are 2 different kinds of additions involved: a *first*-level addition "$+_1$" and a *second*-level addition "$+_2$", hence the term is reflectionally: "{1 $+_1${{1$+_2$1}}".

From the standpoint of a *theory of reflection* (Gunther) which takes into account the *difference* of the levels of reflection (distinction), it would be more interesting to study the reflectional properties of the *process* of "nivilation" of partitions towards a conglomerate of indistinguishable elements instead of the *results* of the elimination of the difference alone. This process second-level reflection is measured by $Ord_{refl}(n)$.

Example

m = (3): {3}, {2 + 1}, {1 + 1 + 1} :first-level partition, P(n)
{3}, {2 + 1}, **{{1 $+_2$ 1} $+_1$1}**, {1 + 1 + 1} : second-level partition, sp_n
$Ord_{refl}(3) = 2$

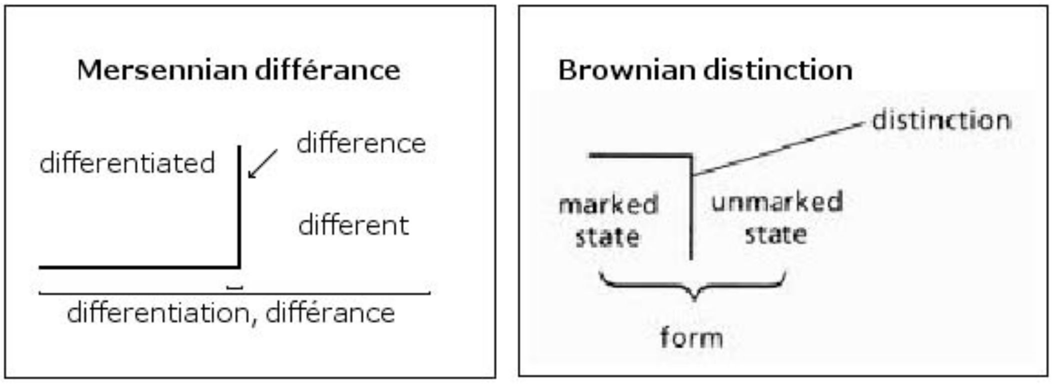
Table

n	$P(n)$	sp_n	diff
1	1	1	–
2	2	2	–
3	3	4	1
4	5	11	6
5	7	30	23
6	11	96	85

Therefore, a calculus with complecity n=2, like the CI and CD, has no properties

of second-level reflection. A calculus with n=3 has 1 second-level property, written as diff(n).
http://memristors.memristics.com/Interplay/Interplay%20of%20Elementary%20Graphematic%20Calculi.html

Diagrams of distinction and differentiation



Metaphors
Brown's *tabula rasa* world, cut by the act of a distinction. Mersenne's streams of signs. Differentiated by the perception of a differentiation.

"Distinction without difference.": ┘ ┘ = ∅ or ⏋| = ∅ but ∅ != ∅ ?
"Difference without distinction."

The motivation for Brownian distinctions are founded in the Mersennian process of differentiation; both are interacting simultaneously together. This interplay of "*constructivist*" and "*recognicist*" actions is marked by the quadralectics of both calculi.

"Discuss the distinction between indicational and differential calculi. Is there a difference or is it a distinction?"
"How to draw a difference between a distinction and a differentiation? How is it indicated?"

At first, the graphematic complementarity approach is in no way forced to establish a *hierarchy* between the concepts and strategies of distinction and differentiation.

Secondly, the involved calculi are not forced to reduce their space of realizations to mono-contexturality.
http://www.cpsa-acsp.ca/paper-2003/gaon.pdf

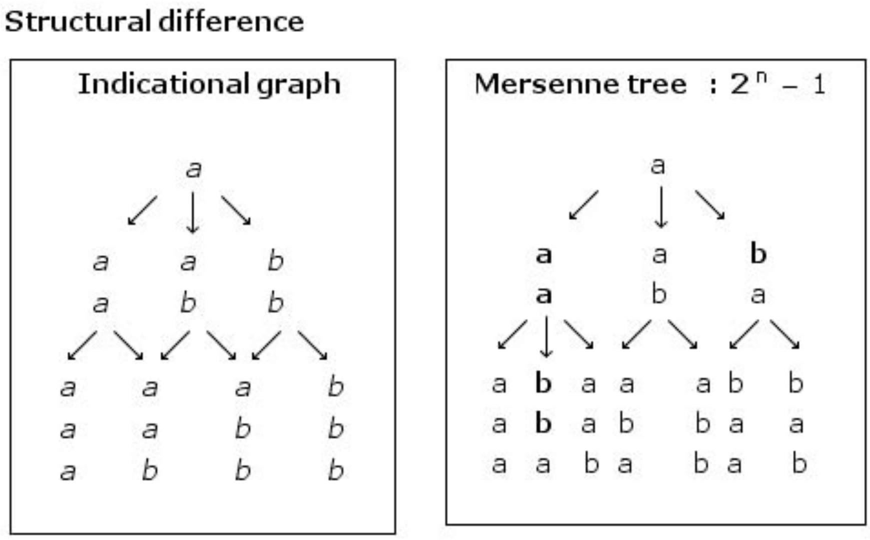
Operator-operand interplay: The "Cross" as an operator and as an operand

"The true nature of the distinction between the pa *on the one hand, and 2 and sentential logic on the other, now emerges. In the latter formalisms,* complementation/negation *operating on "nothing" is not well-formed.*
But an empty Cross *is a well-formed pa expression, denoting the Marked state, a primitive value. Hence a nonempty* Cross *is an* operator, *while an empty Cross is an* operand *because it denotes a primitive value.*
Thus the pa *reveals that the heretofore distinct mathematical concepts of*

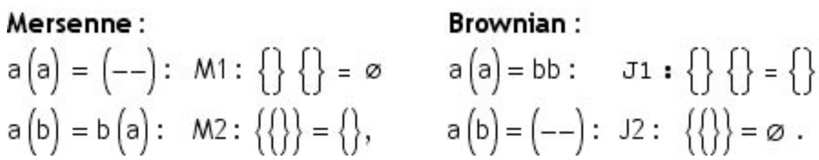
operator *and* operand *are in fact merely different facets of a single fundamental action, the making of a distinction."* (WiKi, Laws of Form)

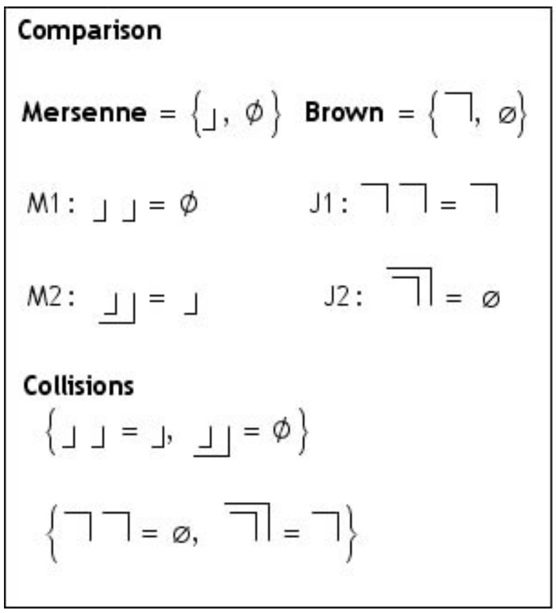
The Cross as an *operand*: ⏋,
The Cross as an *operator*: $\overline{p}$⏋.

Obviously, the *functionality* of the Cross is depending on the contextual use of the Cross. It is lazy thinking to tell that the Cross *is* both an operator and an operand, i.e. *"merely different facets of a single fundamental action"*. The Cross is acting, functioning or playing *as* an operator or (exclusively) *as* an operand. But the Cross *is* not acting as both at once.
A similar situation is well known for the calculus of *Combinatory Logic* (Schönfinkel, Curry, Fey, Rosenbloom, Barendregt).



Bracket interpretation of graphematic constellations





Interpretations, again
LoF
Following Wolfram's statement, according to M. Schreiber:
"A kind of *form* is all you need to compute. A system can emulate rule 110 if it can distinguish: *'More than one is one but one inside one is none.'*
Simple distinctions can be configured into forms which are able to perform universal computations."

LoD
"A kind of structuration is part of what you need to transpute. A system can inscribe the complementary rule 110" if it can differentiate: *'More than one is none but one inside one is one.'*
Complex differentiations can be inscribed into structurations which are able to create pluri-versal transputations."

LoF & LoD
Both together, and set into a polycontextural framework, are staging polyversal co-creations in the doman of distinctions, differentiations and worlds.

2.1.5. Meta-theorems

CI-Brown
T9: *"If any space contains an empty cross, the value indicated in the space is the marked state."* (Varela, p. 114)

Theorem T9:
$e = p_1 \urcorner p_2$
$$e = e_1 \urcorner e_2 \Longrightarrow e_1 \urcorner e_2 = \urcorner.$$
This is properly marking the *context - free* occurrences of distinctions in the CI.

Theorem T12:
$\overline{\overline{p}\rceil p}\rceil = \varnothing$

Theorem T13:
$\overline{\overline{p\,r}\rceil\ \overline{q\,r}\rceil}\rceil = \overline{\overline{p}\rceil\ \overline{q}\rceil}\rceil\ r$
Null

CD-Mersenne
DT9: *"If any constellation is enveloped by a singular differentiation, the situation differentiated in the constellation is the singular differentiation."*

Theorem DT9:
$e = \underline{p_1\ p_2}\rfloor$
$$e = \underline{e_1\ e_2}\rfloor \Longrightarrow \underline{e_1\ e_2}\rfloor = \rfloor.$$
This is properly marking the *context - free* occurrences of differentiations in the CD.

Theorem DT12:
$\underline{\underline{p}\rfloor p}\rfloor = \rfloor.$

With substitution and M1, M2 :

1. p = ⌋: ⌋⌋⌋⌋ = ⌋⌋⌋ = ⌋

2. p = ∅ : ∅⌋∅⌋ = ⌋⌋ = ⌋.

Theorem DT10

The simplification of an expression is unique.

Theorem DT13

Let p, q, r stand for any expressions. Then in any case,

DT13: p r⌋ q r⌋⌋ = p⌋ q⌋⌋ r⌋

Proof :

Let r = ⌋:

p⌋⌋ q⌋⌋⌋ = ⌋⌋⌋ ⌋⌋⌋⌋ = ⌋ ⌋⌋ = ⌋ : DT9, M1, M2

and

p⌋ q⌋⌋ r⌋ = p⌋ q⌋⌋ ⌋⌋ = ⌋⌋ ⌋⌋ = ⌋⌋⌋ = ⌋.

Let r = ∅

p∅⌋ q∅⌋⌋ = p⌋ q⌋⌋ = ⌋ : DT9, M2

and

p⌋ q⌋⌋ ∅⌋ = p⌋ q⌋⌋⌋ = p⌋ q⌋⌋ = ⌋.

2.1.6. Duality

Duality is well known as an interesting property of formal systems with structural and economic advantages; *"Two for one."* (Herrlich)

Duality for indication calculi

P, A ∈ CI ,

P: property and op= duality operation

1. $(A^{op})^{op}$ = A,
2. $P^{op}(A)$ holds if and only if $P(A^{op})$ holds.

Example

X, Y ∈ CI:

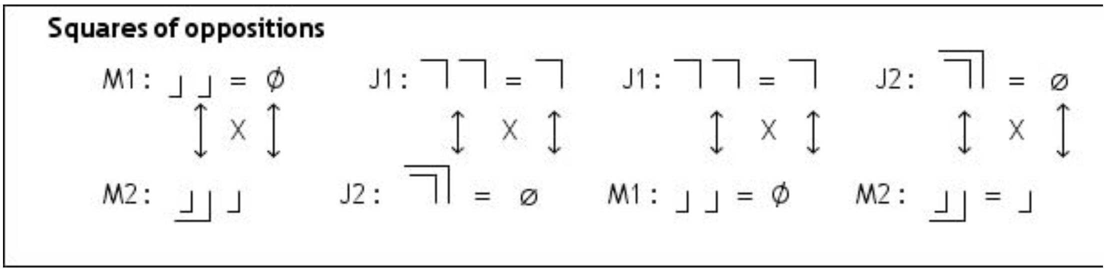
X = Y --> $X^{op} = Y^{op}$

For X = AB, Y = BA:

AB = BA --> A⌉B⌉⌉ = B⌉A⌉⌉

Duality of Mersenne calculi

CM: p = q --> p⌋ = q⌋



2.1.7. Epistemological complementarity

Both characterizations are referring to the distinctions (differentiations) made, and inscribed, by its marks. There is no reference to an outside world which has to be divided and marked needed.

Mersenne calculi are not referring to a world to be divided but to the media they are inscribed. Hence, they are reflecting and referring to themselves instead of an external world which needs observer who are making distinctions in this world.

Also the basic "beginnings" (initials, axioms) of the complementary Brownian and the Mersennian calculi look intriguingly simple and more like duals of each other than highly different complementary approaches, the consequences of the differences and similarities becomes surprisingly decisive and clear in the "algebra" of both calculi.

Spencer-Brown
The most fundamental activity is to draw a distinction and to mark it.
This marks the *constructivitic* appraoch of a subject to the world (Kant).

"Draw a distinction! Mark it!" refers to a so called world, and implies an actor of the distinction.
"*Distinctions splits the world into two parts, [...]. One of the most fundamental of all human activities is the making of distinctions.*" (Principles of Biological Autonomy, Varela, 1979, p. 84)

There is no need to privilege a starting point as the ultimate beginning and determination of human activities as the Brownians are insisting. There are other possibilities too, and there is also no chance to legitimate or to prove the correctness of such a statement about any beginnings.
http://www.vordenker.de/ics/downloads/logik-second-order.pdf

Mersenne
The most fundamental activity is to separate and to identify the separated.
"Without separation no identification, and *vice versa*."

This hints to the transcendental-phenomenological turn to the "*Sachen selbst*" (Husserl, Heidegger) as a complementary approach to cognitivistic constructions.

There is no need to privilege a starting point as the ultimate beginning and determination of human activities as the Brownians are insisting. There are other possibilities too, and there is also no chance to legitimate or to prove the correctness of such a statement about any beginnings. There is also no need to insist on a Mersenne complementary statement about the importance of beginnings.

Spencer-Brown/Mersenne
The *constructivistic* approach of the CI is forgetting that it needs something to construct something, even if this something is self-referentially constructed by itself.
The "*transcendental-phenomenological*" approach of the CD is forgetting that it needs methods to detect something that is not yet a detected method.

Both approaches are denying their "*blind spot*", unmasked by the graphematical approach as the graphematic "*gaps*" of the calculi. Both gaps are complementary to each other.

Instead of the constructivist activity to draw a distinction in a presumed world as demanded by the CI, the CD ask to accept the perceived or encountered difference in the world as a separation or differentiation of something, especially something written or inscribed, independent of an active and constructivist observer but depending on someone who is able to accept what is given. The given is not a pre-given entity from nowhere but a cultural event of other cultural events, i.e. cultural agents. With this turn, miseries of solipsism are results of blindness towards what is given as encountered in an happenstance of encounter.

In other words, the CI is drawing distinctions, the CD is encountering events.
For the CI, the 'patron' might be Immanuel Kant, for the CD, a reference to Alfred North Withehead might be accurate.

Hence, the interpretation of the initials of the CD are becoming:
M1: The iteration (of the acceptance) of an event is the absence of (an acceptance of) an event. And,
M2: The (acceptance of an) event of an event is (the acceptance of) an event.

Graphematics
The most fundamental activity is to live the "Schied" of the "Unterschied", i.e. the 'tinction' of the dis-tinction, distinction and difference, between the two complementary 'tinctions' of Brown and Mersenne and their dual forms of inscriptions as part of the general system of graphematics. The "Schied" of the "Unterschied" is in the history and ressemblance with the process of "différance" as close as possible to the movement of identity.

Both, the Brownian calculus of indication and and the Mersennian calculus of differentation might be based on the graphematic system of inscription. Both appears as interpretations of different graphematic systems, belonging to the general system of graphematics.
How is the system of graphematics introduced? Is there any transparency or is it as obscure as the introduction of the Calculus of Indication?

Graphematics are an interpretation based on the classification of partitions of a set of signs. It is proved that this classification system of sets of signs is complete. With the *Stirling Turn*, graphematics is based on kenogrammatics and not anymore exclusively on semiotics.

Gaps
Gaps appear in the interaction between different calculi, i.e. CI, CD and semiotics. There is no direct access for a calculus to its own gap. Hence, a gap is a *blind spot* of a calculus. An *interactional* calculus of indication and differentiation is including the interactivity of calculi and gaps. Gaps are a third category to the "mark", "unmark" , ∅, and differentiation and absence of differentiation, ∅.

Algebraic and Co-algebraic characterizations
There is also no single point of beginning in graphematical scriptures, like "*Mark it!*". The "algebraic" distinctions of syntax, semantics and pragmatics are not guiding for graphematic formal systems (languages, i.e. scriptures).

Algebraic CI:
"In the beginning there is a space, normally a plane surface, that is featureless but upon which symbols (a primitive notion) may be inscribed." (Meguire)

Co-algebraic graphematics
"There are always symbols to encounter that are defining a space of inscriptions. Such a scriptural space is never featureless, and symbols, marks and signs (complex kenoms) are always distinguishable and are always further differentiated by coordinated inscriptions." (paraphrase)

"Interaction of actors has no specific beginning or end. It goes on forever. Since it does so it has very peculiar properties." (Gordon Pask, 1996)

What might be presupposed for the CI as minimal conditions, is a 2-dimensional open field of kenomic marks. Then, to each identified kenomic mark of the two dimensions the questions leading to the initials "J1" and "J2" might be asked and answered by the indicational rules of J1 and J2 of the Laws of Form.

Calculi, like the CI and the MC, arise as "interpretations" of graphematic streams of kenomic inscriptions.

A formalization of the behavior of the graphematic systems implied by the calculi CI and CD would have to be realized as an *interplay* between algebraic and co-algebraic tectonics.

Also both calculi are, in their complementarity, very similar, they are based on strictly distinct graphematic systems.

The whole wording introduced by GSB to characterize the process of distinction and indication related to a space and time, is a relict from a subordination of writing to the aim of representation in the tradition of sign-related ontology. From a graphematic point of view, only the inscriptions and their laws are of relevance.

This is a step further towards a graphematic understanding of calculi compared to the sketch presented by the *Diamond Calculus* approach which is still emphasizing the proemiality of "world" and "distinction".

2.1.8. Qualitative characterizations

Qualitative characterizations of the graphematics of Brownian and Mersennian configurations
Following Matzka's charcterizations of semiotic, indicational and trito-systems (or strings), but omitting Mersenne systems, the formal approach of Schadach gets some intuitive support additionally to its strictly combinatorial treatement.

General assumption
(A) If the two given tokens of strings have different lengths, then they are different.
If they have equal lengths, then go to (B).

x = (a_1 a_2 a_3... a_n),
y = (b_1 b_2 b_3 ... b_m),
m=n or m!=n.

Identity
(B) For each position i from 1 to the common length, check whether the atom at the i-th position of x equals the atom at the i-th position of y. If this is true for all positions i, then the given tokens are equal, otherwise they are different.

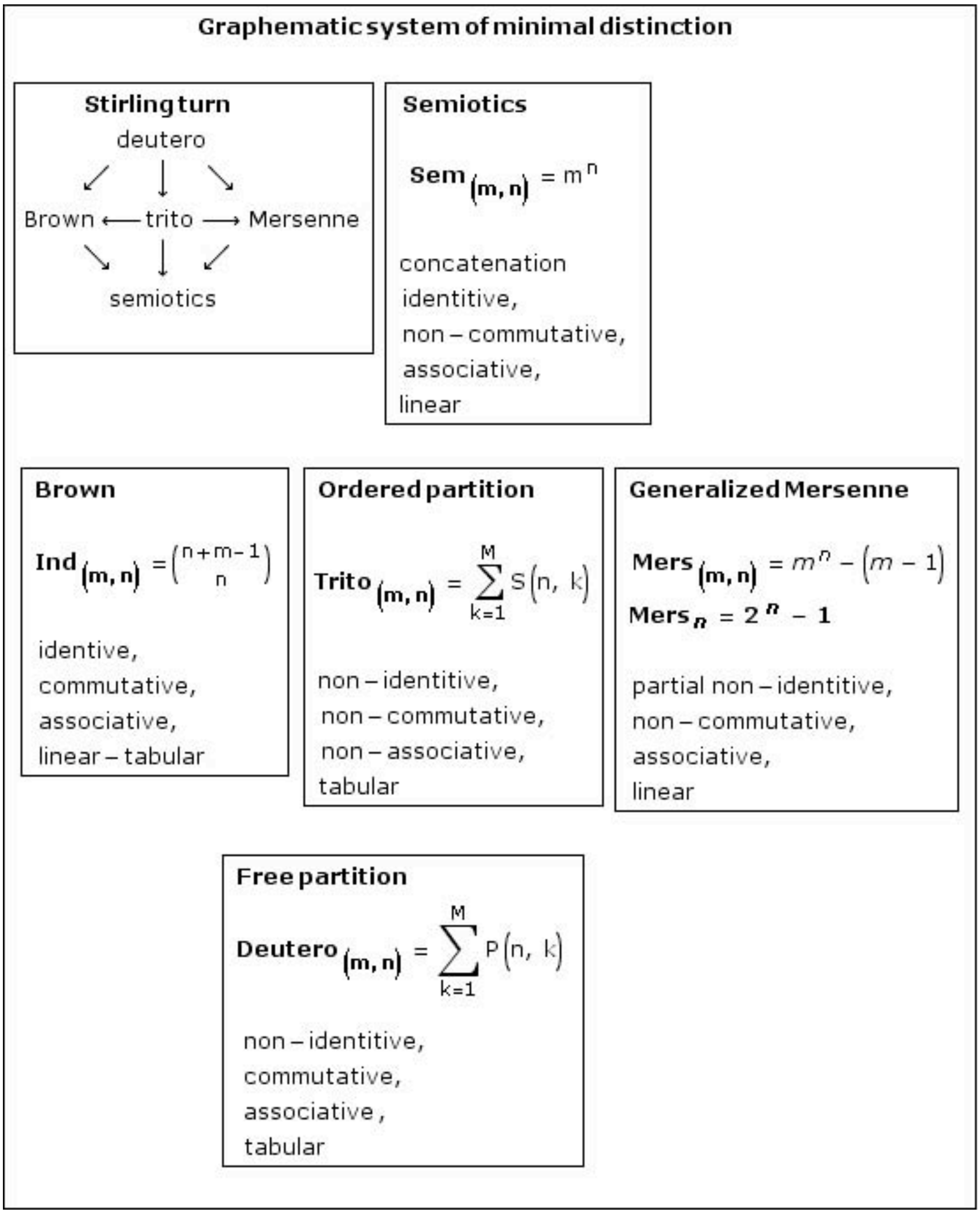
Spencer-Brown
(B') Check whether each atom appears equally often in both string-tokens. If this is the case, then they are equal, otherwise they are different.

Here we have enlarged the abstractive distance between string-token and string-type, by including the abstraction from the order of the atoms into the abstraction from token to type.

Trito-stucture
(B'') For each pair i, k, i<k, of positions, check whether within x there is equality between position i and k, and check whether within y there is equality between position i and k. If within both x and y there is equality, or if within both x and y there is inequality, then state equality for this pair of positions, otherwise state inequality for this pair of positions. If for each pair of positions there is equality, then x and y are equal. Otherwise they are not.

Mersenne
(B''') For each pair i, k, i<k, of positions, check whether within x there is equality between position i and k, and check whether wihin y there is equality between position i and k. If within both x and y there is equality for all i, k, and additionally (B) holds, then x and y are equal. Otherwise they are not.

2.1.9. Combinatorial characterizations



2.2. Comparison of primary arithmetics and algebras of CI and CD

The graphematic complementarity of CI and CD is reflected in the complementarity of their arithmetics and algebras.

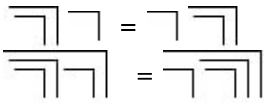
Graphematic constellations

types\values	**aa**	**ab**	**ba**	**bb**
Boolean	aa	ab	ba	bb
Mersennian	aa	ab	ba	–
Brownian	aa	ab	–	bb
Stirling trito	aa	ab	–	–

types | *values*	**aa**	**ab**	**ba**	**bb**
Boolean **neg**	bb	ba	ab	aa
Mersenne **perm**	aa	ba	ab	∅
Brownian **inv**	bb	ab	∅	aa
Stirling	aa	ab	––	––

How are the graphematic situations mirrored in the formulas of the calculi?

For the CI it seems easy to show the property of *commutativity* of the terms, like



How is the *non-commutativity* in the CM represented?

┘┘┘ = ┘ ┘┘ --> ┘ ┘ = ┘ ┘ --> ∅ = ∅.

States and constellations:

CD = {┘ , ∅}

CD-constellations = {┘ ┘, ┘┘ ┘, ┘ ┘┘}.

CI = {┐, ∅}

CI-constellations = {┐┐, ┐ ┐┐, ∅∅}

Concatenation, combination and superposition

CI:

X ∈ CI, Y ∈ CI => X^Y ∈ CI : concatenation

X ∈ CI, Y ∈ CI => X(Y) ∈ CI : superposition

CD:

X ∈ CD, Y ∈ CD => X*Y ∈ CD : combination

X ∈ CD, Y ∈ CD => X(Y) ∈ CD : superposition

Equality in CI and CM

Equality in the CI is defined in a traditional way, i.e. an equality relation is reflexive, symmetric transitive.

Therefore, x = x and x != $\overline{x}$┐, hence:

CI: ┐ != ┐┐ --> ┐ != ∅.

This corresponds in the CI properly the graphematic situation: (aa) != (bb), confirmed with:

CI: ┐┐ != ┐┐ ┐┐ --> ┐ != ∅ ∅ --> ┐ != ∅ .

The corresponding situation for the CD is given with (aa) = (bb) and confirmed with:

CD: ┘ ┘ = ┘┘ ┘┘ --> ∅ = ┘ ┘ --> ∅ = ∅.

And in particular: (a) = (b):

CD: ┘ = ┘┘ --> ┘ = ┘.

Trans-classical concept of equality

As introduced in earlier papers, the *equality* relation gets a deconstruction into a system of different types of equality:

equality, equivalence, similarity, bisimilarity and metamorphosis. All those concepts or paradigms of "equality" are applicable to a general theory of graphematically based calculi.

http://memristors.memristics.com/Church-Rosser%20Morphogrammatics/Church-Rosser%20in%20Morphogrammatics.pdf

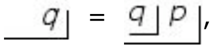
2.3. Basic interpretations

types | *values*	**aa**	**ab**	**ba**	**bb**
Mersenne **perm**	aa	ba	ab	∅
Brownian **inv**	bb	ab	∅	aa

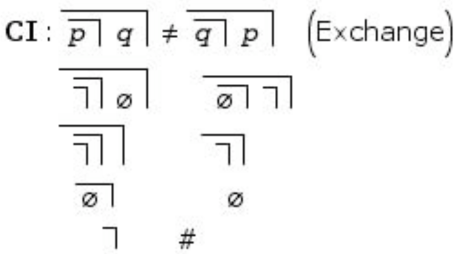
CI : $\overline{pp}$┐ = qq, corresponds to: (aa) $!=_{Brown}$ (bb).

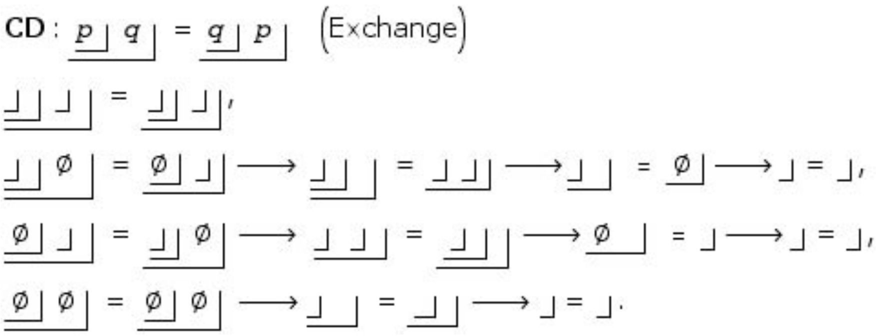
CD: p┘ = ┘, corresponds to: (aa) $=_{Mers}$ (bb).

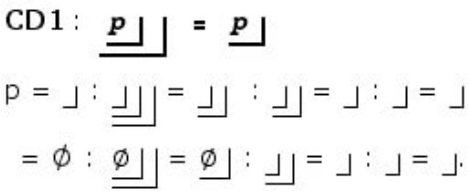
CI: $\overline{p}$┐┐ = p, corresponds to: (ab) $=_{Brown}$ (ba).

CD: , corresponds to: (ab) $!=_{Mers}$ (ba).

Comparison of the CI and CD formula " exchange "





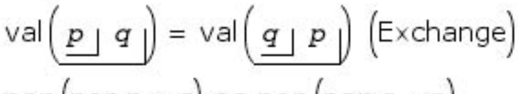


CD2 : ⌋⌋p **= p :**

P = ⌋ : ⌋⌋⌋ = ⌋

= ∅ : ⌋⌋∅ = ∅.

Logic



non(non p ∨ q) *eq* non(non q ∨ p)

non(non(1) ∨ 2) eq non(non(2) ∨ 1)

non(2 ∨ 2) eq non(1 ∨ 1)

1 eq 2 : #

Hence, the formula of "exchange" in the calculus of differentiation CD is *specific* for the CD. It doesn’t hold in the calculus of indication, CI, nor in a Boolean logical interpretation with negation and conjunction (disjunction). Therefore, the calculus of differentiation is not modeled by a Boolean algebra as this is the case for the calculus of indication (minus some differences).

"Negation” in CD?

How to define “negation” in CD if the calculus of differentiation, CD, is negation-invariant?

$\mathrm{val}(\rfloor \rfloor p) = (p)$

$\rfloor \rfloor p = p$

$\rfloor \rfloor \rfloor = \rfloor$

$\rfloor \rfloor \emptyset = \emptyset$.

$\mathrm{val}(\rfloor \rfloor p) = \mathrm{val}(\rfloor p)$: $\mathrm{neg}(p)$

$\rfloor \rfloor p = \rfloor p$

$\rfloor \rfloor \rfloor = \rfloor, \rfloor \rfloor = \emptyset$: #

$\rfloor \rfloor \emptyset = \emptyset, \rfloor \emptyset = \rfloor$: #

$(\rfloor \rfloor)\rfloor = \emptyset\rfloor = \rfloor$,

$\rfloor (\rfloor \rfloor) = \rfloor \emptyset = \rfloor$.

But $(\rfloor \rfloor \rfloor) \neq \emptyset$.

Negation and **self – quotation** modeled by

$\underline{\boldsymbol{p}\rfloor}p$

$p = \rfloor \Longrightarrow \underline{\rfloor\rfloor}\rfloor = \rfloor\rfloor = \emptyset$

$p = \emptyset \Longrightarrow \underline{\emptyset\rfloor}\emptyset = \rfloor$.

With $\underline{\boldsymbol{p}\rfloor}\mathbf{p} = \rfloor\mathbf{p}$:

Negation : $\rfloor\mathbf{p}$

$p = \rfloor \Longrightarrow \rfloor\rfloor = \emptyset$

$p = \emptyset \Longrightarrow \rfloor\emptyset = \rfloor$.

Oscillation

$p = \underline{p\rfloor}p$.

$p = \underline{p\rfloor}p \Longrightarrow$ #.

Oscillation : $\mathbf{p} = \rfloor\mathbf{p}$:

$p = \rfloor \Longrightarrow \rfloor\rfloor = \emptyset$

$p = \emptyset \Longrightarrow \rfloor\emptyset = \rfloor$.

Again,
M1: *A repetition of a quotation is the absence of a quotation.*
M2: *A quotation of a quotation is a quotation.*

3. Primary arithmetics and algebra of CI and CD

3.1. Table of some complementary laws

CI – Meta	**Spencer – Brown CI**	**Mersenne CD**	**CD – Meta**
Arithmetics Condensation Cancellation Order, I1 Number, I2	⌉⌉ = ⌉ $\overline{⌉}\rceil = \varnothing$	⌋⌋ = ∅ $\underline{⌋}\rfloor = ⌋$	Number, M1 Order, M2
Algebra Position, J1	$\overline{\overline{p}\rceil\, p\rceil} = \varnothing$	$\underline{\underline{p}\rfloor\, p\rfloor} = ⌋$	Contra – Position N1
Transposition, J2	$\overline{\overline{\overline{p\,r}\rceil\ \overline{q\,r}\rceil}\rceil} = \overline{\overline{\overline{p}\rceil\ \overline{q}\rceil}\rceil\, r}\rceil$	$\underline{\underline{\underline{p\,r}\rfloor\ \underline{q\,r}\rfloor}\rfloor} = \underline{\underline{\underline{p}\rfloor\ \underline{q}\rfloor}\rfloor\, r}\rfloor$	Contra – Transposition N2
Meta – Theory **Elementic** Duality	$p = q \rightarrow \overline{p}\rceil = \overline{p}\rceil$	$p = q \rightarrow\ ⌋p = ⌋q$	Duality
Simplification Completenes Theorem 11.9	uniqueness provable $e_1 ⌉ e_2 = ⌉$	uniqueness provable $\underline{e_1\ e_2}\rfloor = ⌋$	Simplification Completeness Theorem *D* 11.9
Morphic Heterogeneous Homogeneous constellations Morphic Th11 .9	$X \in \mathrm{Het} \Longleftrightarrow X = X$ $X \in \mathrm{Hom} \Longleftrightarrow X = ⌉X$ $X \in \mathrm{Het} \Longrightarrow \overline{X}\rceil =_{\mathrm{CI}} X$ $X \in \mathrm{Hom} \Longrightarrow \overline{X}\rceil \neq_{\mathrm{CI}} X$ $e_1 ⌉ e_2 = e_2 ⌉ e_1$	$X \in \mathrm{Het} \Longleftrightarrow X = X$ $X \in \mathrm{Hom} \Longleftrightarrow X = ⌋X$ $X \in \mathrm{Het} \Longrightarrow ⌋X \neq_{\mathrm{CD}} X$ $X \in \mathrm{Hom} \Longrightarrow ⌋X =_{\mathrm{CD}} X$ $e_1 ⌉ e_2 = e_2 ⌉ e_1$	Classification of constellations as Hom and Het

CI – Theorems	Spencer – Brown CI	Mersenne CD	CD – Th
Reflexion C1	$\overline{\overline{p}\rceil}\rceil = p$	$\underline{\underline{p}\rfloor}\rfloor = \rfloor$ $\underline{p}\rfloor = \rfloor$	Inflexion
Generation C2	$\overline{pq}\rceil\, q = \overline{p}\rceil\, q$	$\underline{pq}\rfloor\, q = \underline{p}\rfloor\, q$	□
Integration C3	$\rceil\, p = \rceil$	$\underline{\rfloor\, p}\rfloor = \rfloor$ $\rfloor\,\rfloor\, p = \rfloor\, \underline{p}\rfloor = p$	□
Occultation C4 I1 Varela	$\overline{\overline{p}\rceil\, q}\rceil\, p = p$	$\underline{\underline{\underline{p}\rfloor\, q}\rfloor\, p}\rfloor = \underline{p}\rfloor$	□
Iteration C5 (Confirmation)	$p\, p = p$	$p\, p = \emptyset$ $\underline{p\, p}\rfloor = \rfloor$ $\underline{p\, p}\rfloor = \underline{p}\rfloor$	Annulation
Echelon Prop 11.22	$\overline{\overline{\overline{p}\rceil\, q}\rceil\, r}\rceil = \overline{p\, r}\rceil\ \overline{\overline{q}\rceil\, r}\rceil$	$\underline{\underline{\underline{p}\rfloor\, q}\rfloor\, r}\rfloor = \underline{\underline{p\, r}\rfloor\ \underline{\underline{q}\rfloor\, r}\rfloor}\rfloor$	Cascade
(aa) ≠ (bb)	$\overline{p\, p}\rceil = q\, q$	$\underline{p\, q}\rfloor = \underline{\underline{q}\rfloor\, p}\rfloor$	(ab) ≠ (ba)
□		$\underline{p}\rfloor\, \underline{p}\rfloor\, \underline{p}\rfloor = \underline{p}\rfloor = \rfloor$ $p\, p\, p = p$	Iteration
Prop 11.21	$\overline{\overline{p}\rceil\, \overline{q}\rceil}\rceil\ \overline{\overline{p}\rceil\, q}\rceil = p$	$\underline{\underline{\underline{p}\rfloor\ \underline{q}\rfloor}\rfloor\ \underline{\underline{p}\rfloor\, q}\rfloor}\rfloor = \underline{p}\rfloor$	□
Prop 11.23 modification	$\overline{\overline{p}\rceil\, \overline{q\, r\, \overline{s\, r}\rceil}\rceil}\rceil =$ $\overline{\overline{p}\rceil\, \overline{q\, \overline{s}\rceil}\rceil}\rceil\ \overline{\overline{p}\rceil\, \overline{r}\rceil}\rceil$	$\underline{\underline{\underline{p}\rfloor\, \underline{q\, r\, \underline{s\, r}\rfloor}\rfloor}\rfloor}\rfloor =$ $\underline{\underline{\underline{p}\rfloor\, \underline{q\, \underline{s}\rfloor}\rfloor}\rfloor\ \underline{\underline{p}\rfloor\, \underline{r}\rfloor}\rfloor}\rfloor$	□
□	□	□	□

3.2. Explanations for Mersenne laws

C1: A differentiation of a differentiation of a situation p is a differentiation: $\underline{\underline{p}\rfloor}\rfloor = \rfloor$.

C2: The differentiation of 2 situations, p and q, combined with the second situation is the differentiation of the first situation p combined with the second situation q: $\underline{pq}\rfloor\, q = \underline{p}\rfloor\, q$.

C3: A differentiation of a differentiation combined with a situation is a differentiation: $\underline{\rfloor\, p}\rfloor = \rfloor$.

C4: A differentiation of a situation p combined with a differentiation of a combination of a situation q combined with the differentiation of the situation p is a differentiation of p: $\underline{\underline{\underline{p}\rfloor\, q}\rfloor\, p}\rfloor = \underline{p}\rfloor$.

C5: The repetition of a differentiation is the absence of a differentiation: p p = ∅.
(The iteration of the same differentiation is not a differentiation.)

C5’: The differentation of the repetiton of a situations p, is a differentiation *per se*: p p ⌋ = ⌋.
: To differentiate and to differentiate and to differentiate again is to differentiate: p ⌋ p ⌋ p ⌋ =
p ⌋, i.e. p p = ∅, but p p p = p, based on M1, i.e. ⌋ ⌋ ⌋ = ⌋.

4. Proofs of some complementary laws

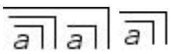
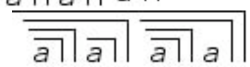
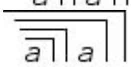
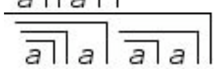
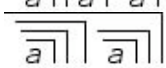
4.1. Algebraic proofs for CI and CD

Algebraic proofs are not referring anymore to states and meta-states of a formula. They refer solely to some selected formulas, proven as correct and used as axioms, and the rules of equality and substitutions applicable in the domain.

4.1.1. Algebraic proof for CI

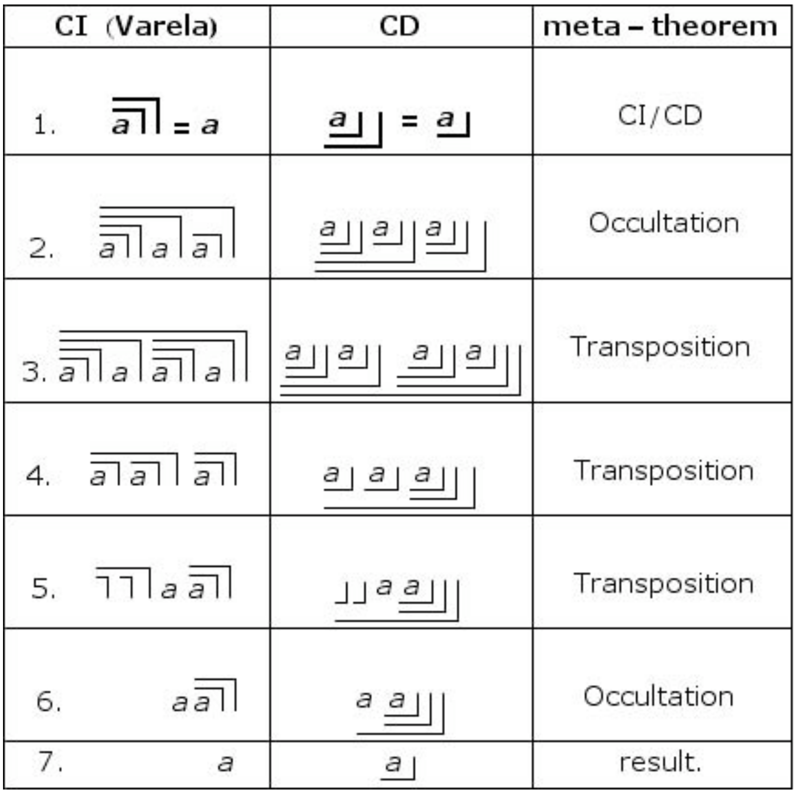
Example
Proof based on (GSB) the CI initials J1, J2 and equality.

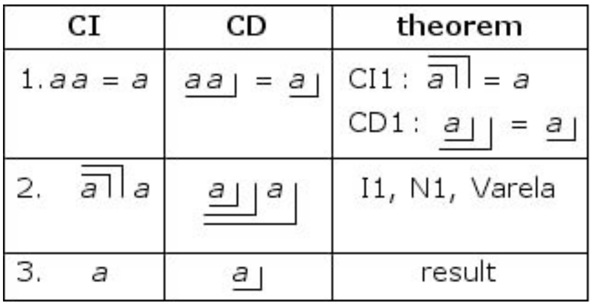
CI1:  **= a**

1.  J1, 0
2.  J2, 1
3.  J1, 2
4.  J1, 3
5.  a J2, 4
6. a J1, 5.

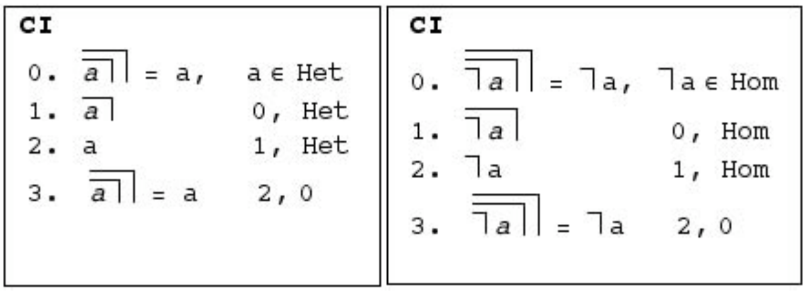
4.1.2. Algebraic proofs for CD

Examples
Proof based on N1, N2 and equality.





Applying the rules Het and Hom



4.2. Tableaux proofs for CI, CD and PC

4.2.1. Tableaux for propositional logic

Rules for concatenation

Concatenation is either disjunctive or conjunctive exclusively.

Rules for signatures

T (¬) = F

F (¬) = T.

Example

log2: $\overline{pq}\rceil q$ = $\overline{p}\rceil q$ disjunctive

1. F $\overline{pq}\rceil q$ (0
2. T $\overline{p}\rceil q$ (0
3. F $\overline{pq}\rceil$ (1
4. F q (1
5. T p q (3
6. T $\overline{p}\rceil$ | T q (2
7. F p | # (6
8. T p | T q | (5
9. # #

log2: $\overline{pq}\rceil q$ = $\overline{p}\rceil q$ conjunctive

1. F $\overline{pq}\rceil q$ (0
2. T $\overline{p}\rceil q$ (0
3. F $\overline{pq}\rceil$ | F q (1
4. T p q | T p q (3
5. T p | T p (4
6. T q | T q (4
7. T $\overline{p}\rceil$ | # (2
8. F p | (7
9. #

4.2.2. Tableaux for the calculus of indication

Rules for concatenation
Concatenation is disjunctive or conjunctive.

Rules for signatures
T (⌉) = F
F (⌉) = T.

e1: $\overline{\overline{p}\rceil}\rceil$ = p

1. F $\overline{\overline{p}\rceil}\rceil$
2. T p
3. T $\overline{p}\rceil$
4. F p
5. #

1′. F p
2′. T $\overline{\overline{p}\rceil}\rceil$
3′. F $\overline{p}\rceil$
4′. T p
5′. #

e2: $\overline{pq}\rceil q$ = $\overline{p}\rceil q$ disj.

1. F $\overline{pq}\rceil q$ (0
2. T $\overline{p}\rceil q$ (0
3. F $\overline{pq}\rceil$ (1
4. F q (1
5. T p q (3
6. T p | T q (5
7. F p | T q | # (2
8. # | # | (5

4.2.3. Tableaux for the calculus of differentiation

Rules for signatures
T (⌋) = T but T (⌋ ⌋) = F
F (⌋) = F and F (⌋ ⌋) = T.

Contradiction set:
= {⌋, ⌋⌋} for CD
value set = {T, F} for signatures.

Rules for concatenation
Concatenation is disjunctive or conjunctive.

Hence,

T p q F p q
/\ |
T p T q F p
F q

Contradiction
1. Proof by contradiction of signatures, based on Mersenne M2: ⌋⌋ = ⌋.
Equality "=" is taken as double implication.

Examples
d1 = $\underline{p⌋}⌋$ = p⌋

Tableau (d1):

1. F p⌋⌋	(0.	1. F p⌋⌋	(0.
2. T p⌋	(0.	2. T p⌋	(0.
3. T p	(2.	3* F p	(1
4. F p⌋	(1.	4* T p	(2
5. F p	(4.	5* #	
6. #			

d2 = p⌋ q⌋ = q p⌋⌋

Tableau (d2):		Stripped of differentiation operation:	
1. F p⌋ q⌋	(0.	1*. F p⌋ q⌋	(0.
2. T q p⌋⌋	(0.	2*. T q p⌋⌋	(0.
3. F p⌋	(1.	3*. F p q	(1
4. F q	(1.	4*. T q p	(2
5. F p	(3.	5*. F p	(3
6. T q | T p⌋	(2.	6*. F q	(3
7. | T p	(6.	7*. T q | T p	(4
7. # #	(6, 4; 7, 5	8*. # | #	

2. Proof of annulation: p p = ∅, based on Mersenne M1: ⌋ ⌋ = ∅

d3: **p p =** ∅

1. F p p	(0.)	1′. F ∅	
2. T ∅	(0.)	2′. T p p	
3. F p	(1.)	3′. T p | T p	
4. F p	(1.)	4′. # | #	(1′, 3')
5. #			

d4: pr⌋ qr⌋ = p⌋ q⌋ r⌋

Tableau (d4):

1. F p⌋ q⌋ r⌋ (0
2. T pr⌋ qr⌋ (0
3. T pr⌋ qr (2
4. F p⌋ q⌋ r (1
5. F p⌋ q (4
6. F r (4
7. F p (5
8. F q (5
9. T pr⌋ **|** Tq r (3
10. T p r **|** T q | T r (9
11. T p | T r **|** (10
12. # # # #

Comment

The formulae $\overline{pq}|q$ = $\overline{p}|q$, p⌋ q⌋ = q p⌋⌋ are generally provable in CI, CD and PC.

4.3. Recursive arithmetics for Mersenne and Brownian calculi

4.3.1. Recursive arithmetics for Mersenne calculi

A comparison of Spencer-Brown's calculus of indication, CI, and the postulated complementary Mersenne calculus of differentiation, CD, is not emphasizing properly enough its *differences* based on its underlying structural difference of the corresponding graphematical systems. In fact, the underlying graph-models, tree and commutative graph, are hinting to the very difference of the structures of both calculi.

It seems that the strictly complementary behavior of both calculi is clearly established. This give the chance to disseminate them in a polycontextural framework, where they hold simultaneously, and allowing to study the complementarity of the indicational and distinctional aspects of events.

A similar comparison to the recursive arithmetic setting is available with the modeling of indicational and differentiational features in a cellular automata framework.

Both calculi are extracting interesting features out of the graphematic systems but are, as far as they are defined up to now, not yet covering the full range of its operative and formal properties. Both calculi, the CI and the CD, are "state"-oriented, i.e. the results of their demonstrations are, in fact, the states "*mark*" and "*unmark*". Both states are atomic, there are no patterns, i.e. morphograms involved. Like it would be suggested by the constellations "(tt), (tf), (ff)" for the CI and "(tt), (tf), (ft)" for the CD. Hence, the variables are defined over tuples of states and not on atomic states.

The asymmetry of both graphematic systems is not yet mirrored in the calculi. It seems, that a further analysis, based on the recursive behavior of both graphematic systems might give some additional insight into the developed structure of the calculi.

In contrast to semiotic and numeric recursivity, i.e. recursivity in the mode of identity, Mersenne and Brown recursivity has to introduce a *normal* form (standard notation) selection from the possible semiotic representations of Mersenne and Brown "strings" or "numbers". Similar to the trito-normal form (tnf) for trito-kenogrammatic operations.

Recursion for Mersenne successor Succ

$a \in \text{Sign} \quad \Rightarrow a \in \text{Mers}$

$x \in \text{Mers}_{hom},\ \text{Succ}(x) \Rightarrow xa,\ xb,\ \overline{x}a \in \text{Mers}$

$x \in \text{Mers}_{het},\ \text{Succ}(x) \Rightarrow xa,\ xb \quad \in \text{Mers}$

Short:

$\text{Succ}(0) = 0 \quad : R1$

$\text{Succ}(x) = \{x\hat{\ }a,\ x\hat{\ }b,\ \overline{x}\hat{\ }a\} \quad : R2.1,\ R2.2,\ R2.3$

$\text{Succ}(x) = \{xa,\ xb\} \quad : R3.1,\ R3.3$

$\overline{x} = (x_i \ldots x_j),\ i = j$

mnf(x): Mersenne normal form of x.

Addition

$\text{Sum}(x,\ o) = x$

$\text{Sum}(x,\ \text{Succ}\,x) = \text{Succ}(\text{Sum}(x,\ y))$

Multiplication

$\text{Prod}(x,\ 0) = x$

$\text{Prod}(x,\ \text{Succ}(y)) = \text{Sum}(x,\ \text{Prod}(x,\ y))$

Examples for Mersenne calculi

Addition Sum

```
Sum(a, 0) = a
Sum(a, Succ 0)  = Succ(Sum(0, a))
                = Succ(a) = {aa, ab, ba}.          : R2.x

Sum(a, Succ a)  = Succ(Sum(a, a))
                = Succ(aa, ab, ba) = {aaa, aab, bba; aba, abb; baa, bab}.

Sum(a, Succ aa) = Succ(Sum(a, aa))
                = Succ(aaa, aab, bba),
                = Succ(aaa) = {aaaa, aaab, bbba},  : R2.x
                = Succ(aab) = {aaba, aabb},        : R2.1, R2.2
                = Succ(bba) = {bbaa, bbab}.        : R2.1. R2.2

Sum(a, Succ aaa) = Sum(a,(aaaa, aaab, bbba)
                 = {aaaaa, aaaab, bbbba; aaaba, aaabb; bbbaa, bbbab}.
```

Multiplication Prod

```
Prod(a, 0) = 0
Prod(a, Succ 0) = Sum(a, Prod(a, 0)) = Sum(a, 0)) = a
                = Prod(a, a) = a
Prod(a, Succ a) = Sum(a, Prod(a; aa, ab, ba)) = Sum(a, (aa, ab, ba))
                = {aaa, aab, bba; aba, abb; baa, bab}.
```

4.3.2. Recursive arithmetics for Brownian calculi

Recursion for Brown successor Succ

$a \in$ Sign		$\Rightarrow a$	$\in$ Brown
$a \in \text{Brown}_{hom}$,	$Succ(a)$	$\Rightarrow \{aa, ab, bb\}$	$\in$ Brown
$x \in \text{Brown}_{hom}$,	$Succ(x)$	$\Rightarrow xa, xb$	$\in$ Brown
$x \in \text{Brown}_{perm}$,	$Succ(x)$	$\Rightarrow \hat{x}a, \hat{x}b$	$\in$ Brown

Short:

$Succ(0) = 0$	: R1
$Succ(a) = \{aa, ab, bb\}$	: R2.1, R2.2, R2.3
$Succ(x) = \{xa, xb\}$	: R3.1, R3.2
$Succ(x) = \{\hat{x}a, \hat{x}b\}$	: R4.1, R4.2

$\hat{x} = (x_i, x_j),\ i \neq j$

bnf(x) : Brownian normal form of x.

Examples for Brown calculi

Addition Sum

```
Sum(a, 0) = a
Sum(a, Succ a) = Succ(Sum(a, a))
               = Succ(aa, ab, bb) = {aaa, aab; abb; bbb} : R2.x
                                   with {aba, bba} ∉ bnf
Sum(a, Succ aa) = Succ(Sum(a, aa))
                = Succ(aaa, aab, bba, bbb)
                = {aaaa, aaab, bbba; aaba, aabb; bbaa, bbab; bbbb}.
                                   with {aaba, bbaa, bbab} ∉ bnf
Sum(a, Succ ab) = Succ(Sum(a, aa))
```

```
                 = Succ(aa, ab, bb) = {aaa, aab; abb; bbb}.

Sum(a, Succ bb) = Succ(Sum(a, aa))
                 = Succ(aa, ab, bb) = {aaa, aab; abb; bbb}.
```

Multiplication Prod

```
Prod(a, 0) = 0
Prod(a, Succ 0) = Sum(a, Prod(a, 0)) = Sum(a, 0)) = a
                 = Prod(a, a) = a
Prod(a, Succ a) = Sum(a, Prod(a; aa, ab, bb)) = Sum(a, (aa, ab, bb))
                 = {aaa, aab; abb; bbb}.
```

Comparision

```
Prod(a, Succ a)
```

Brown: `Sum(a, Prod(a; aa, ab, bb)) =`
`Sum(a, (aa, ab, bb)) =`
`{aaa, aab; abb; bbb}.`

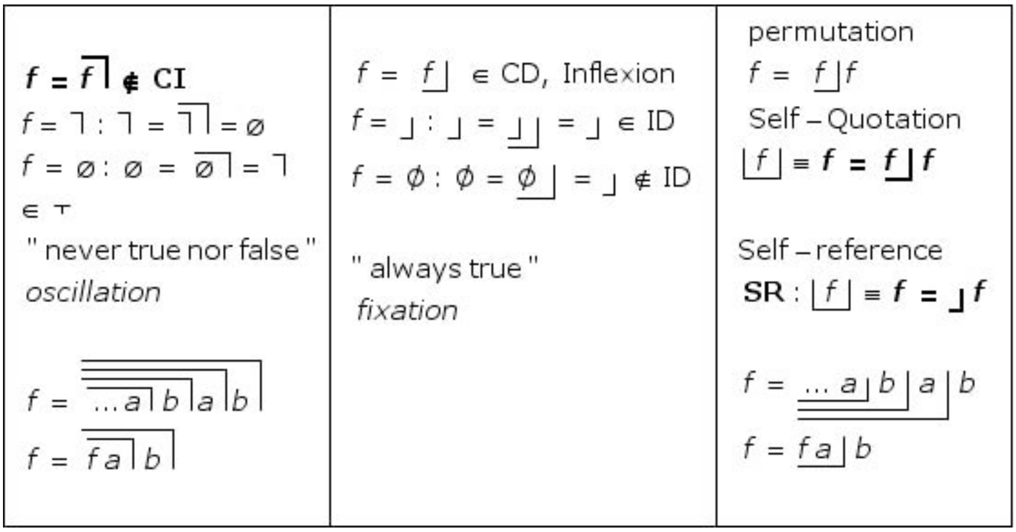
Mersenne: `Sum(a, Prod(a; aa, ab, ba)) =`
`Sum(a, (aa, ab, ba)) =`
`{aaa, aab, bba; aba, abb; baa, bab}.`

5. Recursion and self-referentiality

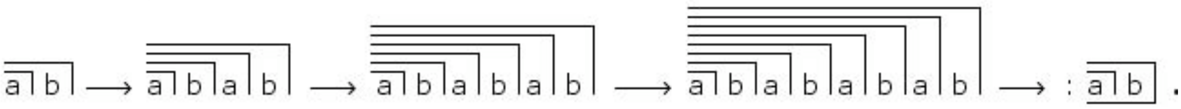
5.1. Reentry for CD

5.1.1. Reentry introduction and comparison

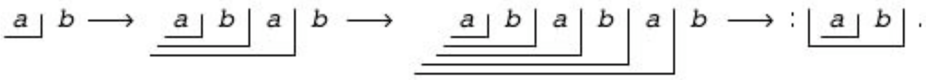
"By adding reentry as a third term, Varela took Spencer-Brown's work a step further and left behind the tame, two-valued world of Aristotelian logic. Varela made the radical assertion that reentry, along with paradoxical dynamics it entails, is built right into the very structure of the form. Varela upheld reentry as the cornerstone to autonomous functioning in nature." Terry Marks-Tarlow, Mario E. Martinez, *Francisco Varela (1946-2001)*, in: Society for Chaos Theory in Psychology & Life Sciences 2001, 9 (2), 3-5.
http://www.biocognitive.com/images/pdf/Francisco_Varela.PDF



LoF example



LoD example



Eigenvalue for differentiations

f = f ⌋ f

f = ⌋ : ⌋⌋ ⌋ = ⌋⌋ = ∅

f = ∅ : ∅⌋∅ = ∅⌋ = ⌋ .

f ⌋ f = ⌋f :

f = ⌋f ∈ Oscillation

|f| ≡ **f** = **f** ⌋ **f** : Self – Quotation

= f f⌋ .

Because of f⌋ f = ⌋f, we get :

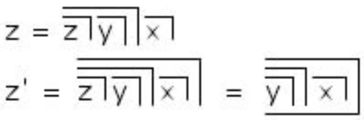
SR : |f| ≡ **f** = ⌋ **f**.

This hints to Smullyan's self – quotation of an expression.
Representation of logical negation in the CD with " f ⌋ f ".

http://memristors.memristics.com/MorphoReflection/Morphogrammatics%20of%20Reflection.html

5.1.2. A flip-flop application:

Brownian case



(Varela)

6. Bifunctoriality of Brownian and Mersennian calculi

6.1. Polycontextural distribution of CI and CD

Brownian and Mersennian calculi are accessible to category-theoretic considerations. And might therefore been distributed over different contextures of a polycontextural category. Interactions between different contextures containing different calculi are well ruled by the construction of bifunctoriality between discontextural contextures. Such a polycontextural dissemination is just a first step to emphazise with the means of category theory the complementary aspects and their interplay of Mersennian and Spencer-Brownian calculi.

CI:

J1: $\urcorner_2 \urcorner_1 = \urcorner_2$

cat-J1:

$$\begin{array}{ccc} & \xrightarrow{\urcorner_1} & \\ \urcorner_3 & \searrow & \downarrow \urcorner_2 . \end{array}$$

$$\mathrm{Mor}(\urcorner_2) \circ \mathrm{Mor}(\urcorner_1) = \mathrm{Mor}(\urcorner_3)$$

J2: $\overline{\urcorner_1 \urcorner}_2 = \varnothing$

cat-J2: $\bullet \leftrightarrows$

$$\overline{\mathrm{Mor}}(\urcorner_1) \circ \mathrm{Mor}(\urcorner_1) = \mathrm{id}$$

CD:

M1: $\lrcorner_1 \lrcorner_2 = \emptyset$

cat-M1: $\bullet \leftrightarrows$

$$\overline{\mathrm{Mor}}(\lrcorner_1) \circ \mathrm{Mor}(\lrcorner_1) = \mathrm{id}$$

M2: $\underline{\lrcorner_1 \lrcorner}_2 = \lrcorner_3$

cat-M2:

$$\begin{array}{ccc} & \xrightarrow{\lrcorner_1} & \\ \lrcorner_3 & \searrow & \downarrow \lrcorner_2 . \end{array}$$

$$\mathrm{Mor}(\lrcorner_2) \circ \mathrm{Mor}(\lrcorner_1) = \mathrm{Mor}(\lrcorner_3).$$

Mediation of Brown, Mersenne, Semiotics

$$\text{Semiotics: } \begin{array}{ccccc} \alpha \equiv \lrcorner & \longrightarrow & \emptyset & : \text{Mersenne} \\ \downarrow & \updownarrow & X & \updownarrow & \\ \square \equiv \varnothing & \longleftarrow & \urcorner & : \text{Brown} \end{array}$$

Null

BIFUNCTORIALITY OF MERSENNE, BROWN and SEMIOTICS

$$\left(\mathcal{U}_1 \bigcap_{1.2} \mathcal{U}_2\right) \bigcap_{1.2.3} \mathcal{U}_3 = \emptyset$$

$$\mathcal{U}^{(3)} = \left(\mathcal{U}_1 \amalg_{1.2} \mathcal{U}_2\right) \amalg_{1.2.3} \mathcal{U}_3$$

$$\mathcal{U}_1 = \left\{\text{Brown}_2,\ \text{distinction}_2\right\}$$

$$\mathcal{U}_2 = \left\{\text{Mersenne}_1,\ \text{differentiation}_1\right\}$$

$$\mathcal{U}_3 = \left\{\text{Semiotics}_3,\ \text{concatenation}_3\right\} :$$

$$\begin{bmatrix} g_{Mers} & - & g_{Sem} \\ f_{Mers} & g_{Brown} & - \\ - & f_{Brown} & f_{Sem} \end{bmatrix} :$$

$$\left(\begin{pmatrix} \left(f_{Mers} \circ^{1.0.0} g_{Mers}\right) \\ \amalg_{1.2.0} \\ \left(f_{Brown} \circ^{0.2.0} g_{Brown}\right) \end{pmatrix} \begin{matrix} \\ \amalg_{1.2.3} \\ \left(f_{Sem} \circ^{0.0.3} g_{Sem}\right) \end{matrix}\right) = \left(\begin{pmatrix} f_{Mers} \\ \amalg_{1.2.0} \\ f_{Brown} \end{pmatrix} \begin{matrix} \amalg_{1.2.3} \\ f_{Sem} \end{matrix}\right) \circ_{1.2.3} \left(\begin{pmatrix} g_{Mers} \\ \amalg_{1.2.0} \\ g_{Brown} \end{pmatrix} \begin{matrix} \amalg_{1.2.3} \\ g_{Sem} \end{matrix}\right)$$

$\amalg$: mediation between contextures

$\circ$: composition of morphisms

= : equivalence

With such a bifunctorial framework it is possible to distribute complementary formulae over the polycontextural grid.
From the point of view of the third contexture, both calculi are accessible to analysis of their semiotic properties and the embedment into the general framework.

6.2. Monocontexturality of CI and CD

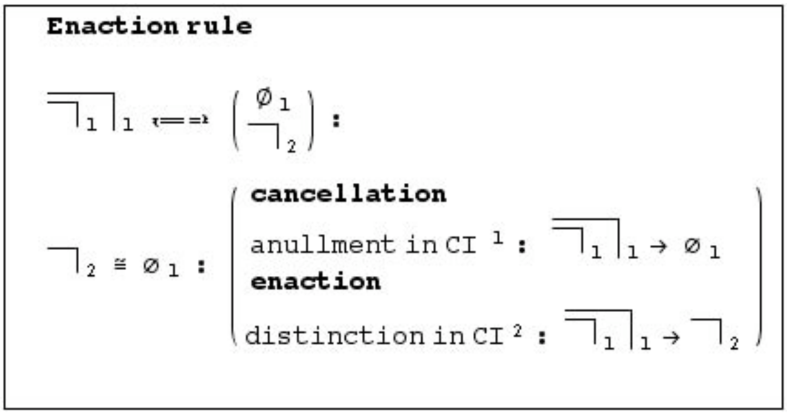
Albeit the insistence of a "two-dimensional" notation and conception of the indicational Laws of Form, any dimensionality is reduced to one-dimensional linearity by the primary arithmetic law J2 and the law of reflexion. The same holds complementarily for the CD. The law of double differentiation is reduced to a single differentiation. Their might be some topological detours but the results of such journeys is always something simple and never involved in any non-reducible complexity. Such a *indicational* and *differentiational* simplicity is, together with the *identificational* simplicity of semiotics (logic), a strong criterion for mono-contexturality.

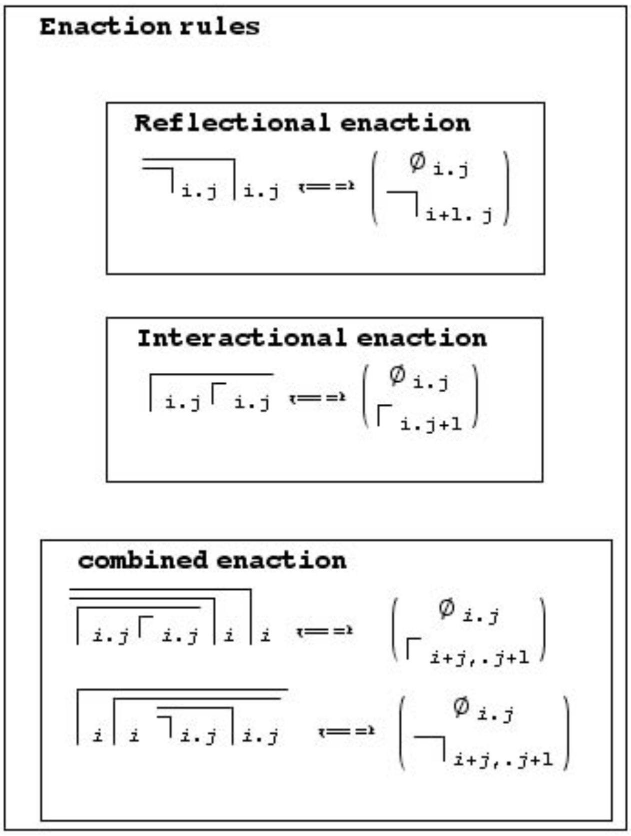
As it was pointed out with the paper "*Diamond Calculus of Formation of Forms"* there is a natural way to disseminate the calculus of indication over different loci: the double meaning of the "zero'-indication as a nullity in its calculus and at once as an indication in a neighboring other indicational calculus. This is formalized by the law of *enaction*. Obviously, the same holds complementarily for the calculus of differentiation too.

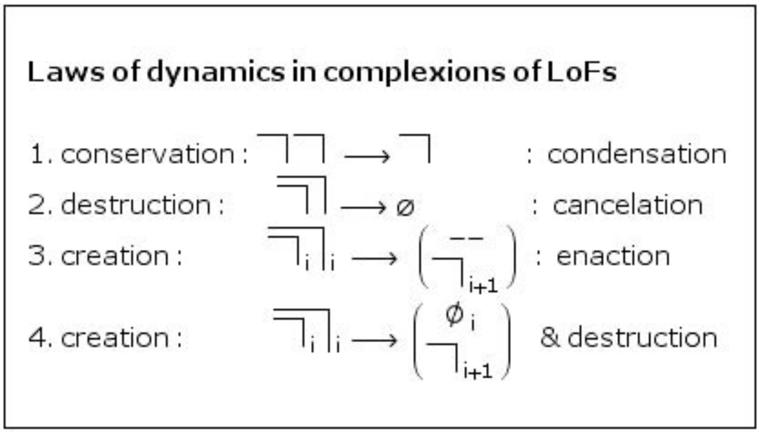
The "*Diamond Calculus*" paper is not yet dealing with the complementary calculus of differentiation, CD. The elaborations for the CI of "*interactional and reflectional*" indications, *enactions* and *retro-grade recursivity* have to be mirrored in the complementary setting of the CD.

With the concept of distributed and mediated enaction, the limitations drawn by the mono-contexturality of the CI and the CD are well overcome. The price to pay is an acceptance of other graphematic systems as grid for dissemination, keno- and morphogrammatics, of the trito-structure of graphematics. This polycontextural approach is not considered in this study.

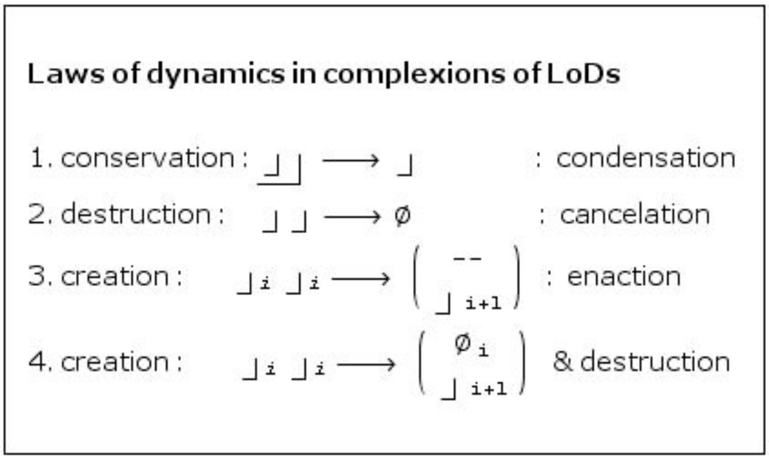
6.2.1. Enaction rules for the CI







6.2.2. Enaction rules for the CD



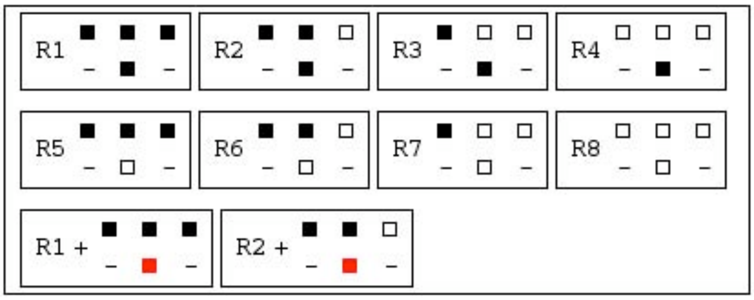
7. Cellular automata of CI and CD

7.1. Indicational CAs

Indicational CAs are identical to the indCAs introduced in previous papers.

Calculus of indication: a=a, a!=b, ab=ba
For 1D indCA:
Indicational normal form (inf):
inf([■ □ ■]) = inf([□ ■ ■])= [■ ■ □])
inf([□ ■ □]) = inf([□ □ ■]) = [■ □ □)]
inf([■ □ ■]) != inf([□ ■ □])
inf([■ ■ ■]) !=inf([□ □ □])

Enactional rule set for indicational cellular automata indCA



Example

indCA, r = {5, 6, 3, 8}

Nr.l	1	2	3	4	**5**	6	7	8	9	rule = {5, 6, 3, 8}
1	□	□	□	□	■	□	□	□	□	3, 3, 3
2	□	□	□	■	■	■	□	□	□	3, 6, 5, 6, 3
3	□	□	■	x	x	x	■	□	□	3, 3, 3, 8, 3, 3, 3
4	□	■	■	■	x	■	■	■	□	6, 5, 5, 6, 6, 6, 5, 5, 6
5	x	x	x	x	x	x	x	x	x	stop

7.2. Mersenne CAs

$2^3 - 1 = 7$: {aaa, aab, aba, abb, bba, bab, baa}, with (aaa) $=_{Mers}$ (bbb).

Mersenne Calculus : a ≠ b, ab ≠ ba, aa = bb.
For 1 – D indCA :
Mersenne normal form (Mers) :

Mers([■ □ ■]) ≠ Mers([□ ■ ■]) ≠ Mers([■ ■ □])

Mers([□ ■ □]) ≠ Mers([□ □ ■]) ≠ Mers([■ □ □])

Mers([■ ■ □]) ≠ Mers([□ ■ □])

Mers([■ ■ ■]) = Mers(([□ □ □]).

$$\text{MersenneCA}^{(3,2)} = (2^3 - 1) \times P(2, 2) = 7 \times 2 = 14.$$

Rules for Mersenne CA $^{(3, 2)}$

R1 ■ ■ ■ / – ■ –	R2 ■ ■ □ / – ■ –	R3 ■ □ ■ / – ■ –	R4 ■ □ □ / – ■ –	R5 □ □ ■ / – ■ –	R6 □ ■ □ / – ■ –	R7	
R8 ■ ■ ■ / – □ –	R9 ■ ■ □ / – □ –	R10 ■ □ ■ / – □ –	R11 ■ □ □ / – □ –	R12 □ □ ■ / – □ –	R13 □ ■ □ / – □ –		

Example

Nr.l	1	2	3	4	**5**	6	7	8	9		rule = rule1 .2 .10 .4 .12 .6 .7
1	□	□	□	□	■	□	□	□	□		12, 6, 4
2	□	□	□	x	■	■	□	□	□		1, 12, 7, 2, 4
3	□	□	■	x	■	■	■	□	□		12, 6, 12, 4, 1, 9, 4
4	□	x	■	x	■	■	x	■	□		1, 12, 6, 10, 7, 2, 10, 6 , 4
5	■	x	■	x	■	■	x	■	■	stop	abstraction : (aaa) $=_{MERS}$ (bbb)

8. Appendix

8.1. Appendix 1: Headaches with complementary calculi

If two formal systems have a very close familiarity as a duality or even a complementarity, and are therefore to some degree nearly indistinguishable, but you nevertheless discovered in a strange situation of an insight a decisive difference between them. Then it might easily be possible, as in my case, that you get nightmares of endless oscillations and manifestations of something you don't yet have access to, and what, as far as you guess, what it could be, you anyway wouldn't like at all.

That's what happens with the discovery of the complementary calculus of indication, a calculus I call a Mersenne calculus of *differentiation* and *separation*, in contrast to the Spencer-Brown calculus of indication and distinction. I have never been a friend of this calculus of *The Laws of Form*, therefore to get involved with its complementary calculus is no pleasure at all.

Obviously, to get rid of the headache with the CI and its ambitious and annoying celebrations, especially in German humanities, the best is to show, or even to prove, that there is a complementary calculus to the calculus of indication, too. And furthermore, both calculi are based on the general graphematic system, where they are located as the two only non-kenomic systems together with the graphematic system for semiotics as the sole pure identity system.

In fact, this is also a very late contribution to clear a situation that has not only produced unnecessary nightmares, I couldn't avoid with my early papers, but also a late confirmation for the importance of Gotthard Gunther's approach in this context that was demolished by the propaganda of the so-called Second-Order Cyberneticians, following Heinz von Foerster's enthusiasm for self-referentiality, fixed points and the famous *circulus creativus* based on a non-existing chapter about the self-referentiality of reentry in George Spencer-Brown's *Laws of Form*.

It turns out that the calculus of indication is the smallest possible calculus of the graphematic indicational system. Because Mersenne and Brownian calculi are conceived as complementary, this restriction holds for the original Mersenne calculus too.

With that, the sectarian propaganda for the CI boils down to a strictly one-sided and utterly restricted endeavour.

In-between I have written some papers dealing with the complementarity and applications of the concepts of the CI and the CD in the framework of a general graphematics.

There might be still too much undeliberated obfuscation involved, at least, some clear aspects of the new calculus of differentiation, CD, and its complementarity to the calculus of indication and distinction, positioned in the system of graphematics, are now elaborated as far as it takes to get a primary understanding of the new situation.

Specification

types	calculus	algorithm	FSM	CA	data type
identification	semiotics	◻	◻	◻	◻
Indication	GSB	◻	◻	◻	◻
Differentiation	Mersenne	◻	◻	◻	◻
kenomic trito	Stirling	◻	◻	◻	◻
kenomic deutero	Pascal	◻	◻	◻	◻

This study *"Interplay of Elementary Graphematic Calculi"* is a direct continuation of the previous paper *"Graphematic System of Cellular Automata"* which is studying 9 levels of graphematical inscription.

8.2. Appendix 2: Logical interpretations

8.2.1. Boolean algebra and CI

"Freilich sind die primäre Algebra und die Aussagenlogik nur bedingt isomorph (strukturgleich). Neben der ungewöhnlichen Verwendung des leeren Ausdrucks, der einer beliebigen Folge von adjunktiv zugefügten Konstanten für >>das Falsche<< entspricht und der >>topologisch invarianten Notation<<, die die Reihenfolge von Adjunktions- und Konjunktionsgliedern irrelevant macht, und damit die entsprechenden Kommutativ- und Assoziativgesetze im Rahmen der primären Algebra überflüssig werden läßt, besteht ein entscheidender Unterschied zu nahezu allen gängigen formalen Systemen in der anadischen Verwendung (das heißt Verwendung ohne Stellenzahl für die Argumente) des Operators ⏋.
Während in der gewöhnlichen Aussagenlogik die Negation einstellig ist und eine Adjunktion als zweistellige Verknüpfung >>a<< und >>b<< zu >>a oder b<< verknüpft, hat der Operator ⏋ jede beliebige endliche Stellenzahl." (Matzka, Varga, Motive und Grundgedanken der >>Gesetze der Form<<, 1993)

Recalling Varela:
CI: Calculus of Indication,
PC: Propositional Calculus,
Variables: A, B, ... ∈ CI, PC.
Procedure: Π.

Definition B.1
```
If A is ¬ B, write B⏋ for A in CI;
If A is B ∨ C, write BC for A in CI;
If |- A in PC, write Π(A) = ⏋in CI;
If |- ¬A in PC, write Π(Ā) = ⏋in CI.
```

Lemma B.2
To every expression in PC there corresponds an indicational form.

Lemma B.3
Every demonstrable expression in PC is equivalent to the cross, ⏋, in CI.
(Varela, Principles of Biological Autonomy, 1979, p.285)

DeMorgan

A⏋⏋ = A
A⏋B⏋⏋ = *AB* corresponds to: ¬ (¬ A ∧ ¬ B) = A ∨ B.

A	*B*	*A*⏋	*B*⏋	*A*⏋*B*⏋	*A*⏋*B*⏋⏋	*A B*	*AB*⏋	*A*⏋*B*
⏋	⏋	⌀	⌀	ϕ	⏋	⏋	⏋	⏋
⏋	⌀	⌀	⏋	⏋	⌀	⏋	⏋	⌀
⌀	⏋	⏋	⌀	⏋	⌀	⏋	⌀	⏋
⌀	⌀	⏋	⏋	⏋	⌀	⌀	⏋	⏋

contra
"I want to conclued by emphazising once again, that the calculi of indication are not a subtle form of logic. They really intent something quite different ..." (Varela, 1979)

Nevertheless, there is a strong isomorphism between the CI and Boolean algebra (Kaehr, Schwartz).
Such a kind of isomorphism is not (easily) to establish between the CD and a Boolean algebra.

8.2.2. Mersenne calculus and algebra

Definition D.1
If A is ¬ B, write ⌋*B* for A in CD;
If A is ¬ B ⟺ ¬ C, write B C for A in CD;
If ⊢ A in PC, write Π (A) = ⌋ in CD;
If ⊢ ¬ A in PC, write Π ($\bar{A}$) = ⌋ in CD.

Lemma D.2
To every expression in PC there corresponds a *differential* form.

Lemma D.3
Every demonstrable expression in PC is equivalent to the mark, ⌋, in CD.

DeMorgan complementarity
¬ A ≅ ⌋A
¬ A ⟺ ¬ B ≅ A B
(⌋*A* ⌋*B*) $=_{Mers}$ A B.
A ⟺ B ≅ ⌋ (A B)

Mersenne CD translates into a " bi – conditional " logical calculus.
Exchange between disjunction and conjunction via negation is not modeled in the CD.
A ∨ B ≅ A B : *wrong*
A ∧ B ≅ ⌋(⌋A ⌋B) = AB : *wrong*

Double " negation "
⌋(⌋ A) = A :
⌋ (⌋ ⌋) = ⌋
⌋(⌋ ∅) = ∅.

Concatenation A B is a contra – valence : A B ≇ A ≇ B
⌋ (⌋*A* ⌋*B*) ≠ AB.
(*A* ⌋*A*) = ⌋.
⌋ (⌋*A* ⌋*B*) = ⌋ (*A B*) = ⌋*A B* = *A* ⌋ *B*.
A B = ⌋*A* ⌋*B* (contravalence).

A	B	⌋A	⌋B	⌋A⌋B	⌋(⌋A⌋B)	AB	⌋(AB)	⌋AB	A⌋B	AB⌋
⌋	⌋	∅	∅	∅	⌋	∅	⌋	⌋	⌋	⌋
⌋	∅	∅	⌋	⌋	∅	⌋	∅	∅	∅	⌋
∅	⌋	⌋	∅	⌋	∅	⌋	∅	∅	∅	⌋
∅	∅	⌋	⌋	∅	⌋	∅	⌋	⌋	⌋	⌋

8.2.3. Trichotomic interpretation

CD : Calculus of differentiation

The *differential* domain of the Mersenne calculus is trichotomic : *Mers* = {tt, tf, ft}.

non (tt) $=_{Mers}$ tt

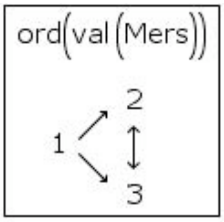
non (tf) $=_{Mers}$ ft

Negation in Mersenne is permutation.

num ({tt, tf, ft}) = (1, 2, 3) = X

non (1, 2, 3) = (1, 3, 2)

Mersenne : ord (X) = 1 < (2, 3),



negation	A	¬ A
Mersenne	1	1
□	2	3
□	3	2

conj	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

¬ A ∨ ¬ B	1	3	2
1	1	1	1
3	1	3	2
2	1	2	2

⟹

¬ (¬ A ∨ ¬ B)	□	□	□
□	1	1	1
□	1	2	3
□	1	3	3

.

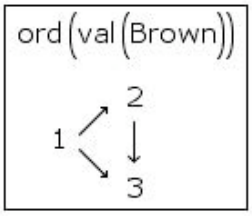
Therefore : ⌋⌋ = ∅ .

CI : Calculus of indication

Spencer – Brown :

Brown = {tt, tf, ff}

ord(X) = 1 < 2 < 3.



Negation

non(tt) = ff

non(tf) = tf, because (ab) $=_{Ind}$ (ba)

Negation in Brown is inversion (negation)

negation	A	¬ A
Brown	1	3
□	2	2
□	3	1

Therefore : ⅂⅂ $=_{Brown}$ ∅.

In contrast: Logical complementary systems

Johannes Oetsch and Hans Tompits, Gentzen-type Refutation Systems for Three-Valued Logics

"In contrast to conventional proof calculi that axiomatise the valid sentences of a logic, refutation *systems, or* complementary *calculi, are concerned with axiomatising the invalid sentences. Hence, the inference rules of such systems formalise the propagation of refutability instead of validity.*

While the traditional method to show that a formula is not valid is exhaustive search for counter models, refutation systems establish invalidity by deduction and thus in a purely syntactic way.

In fact, already the forefather of modern logic, Aristotle, studied rules that allow to reject assertions based on already rejected ones."

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8.2.4. Constellations

Combinatorial studies are determining the number and structure of indicational and differentiational constellations. Constellations are similar to logical functions for Boolean algebras and propositional logics.

The hidden dynamics of such constellations are becoming manifest and productive if the constellations are interpreted as rules of cellular automata or finite state machines.

The classical presentation of the *Laws of Form*, and its calculus of indication, are not giving any hint to dynamize the very structure of the calculus. Dynamics are studied inside the calculus on the base of so called second-order formulas and reentry functions based on speculations about recursion in the framework of the CI (Kauffman, Varela, et al).

The new turn, presented in previous papers, is changing the static constellations into dynamic CA rules. Of special interest is the functional change of morphograms into morphic CA rules. But the same mechanism works for indicational and differentiational calculi too.

Where are the constellations from? Constellations are automatically introduced with the use of *variables* for *values* and to build *expressions* by concatenation of terms. Therefore, the study of constellations instead of singular values (states) as results or cases of demonstrations (proofs) is well defined.

Following the classification of the valuation of constellations by Spencer-Brown, there are, as for propositional logic too, 3 different classes: *truth, untruth, or contingency*.

The first we learn from the constellations of the values $\{\urcorner, \varnothing\} \in$ CI is that the correspondence between the CI and Boolean logic, with a distribution of $\binom{n+m-1}{n}$ for the CI and m^{m^2}, m=2, for Booleans, is not as close as it seems. Neither for the CD, with $2^n - 1$.

Hence, the story of the *isomorphism* between the CI and Boolean algebra has to be reconsidered again.

Indicational case

CI:

Brown : $\{\urcorner, \varnothing\}^2{}_{ind} \longrightarrow \{\urcorner, \varnothing\}$:

$$\text{Ind}_{(n,m)} = \binom{n+m-1}{n} : \binom{4+2-1}{4} = \binom{5}{4} = 5, \text{ for } m = 2,\ n = m^2 = 4.$$

Reduction quotient:

$$\text{reduction}\left(\frac{\text{Logic}(m, n)}{\text{Brown}(m, n)}\right) = \frac{m^{m^n}}{\binom{n+m-1}{n}} : \frac{2^{2^2}}{\binom{4+2-1}{4}} = \frac{16}{5} = 3,3$$

Null

CI standard normal fom

[op1]	[op2]	[op3]	[op4]	[op5]
⏋	⏋	⏋	⏋	∅
⏋	⏋	∅	⏋	∅
⏋	⏋	∅	∅	∅
⏋	∅	∅	∅	∅

Logical representation of indicational constellations

Following the example of G.Spencer-Brown, a CI-expression is representing a set of logical functions (Laws of Form, 1979, p. 115).

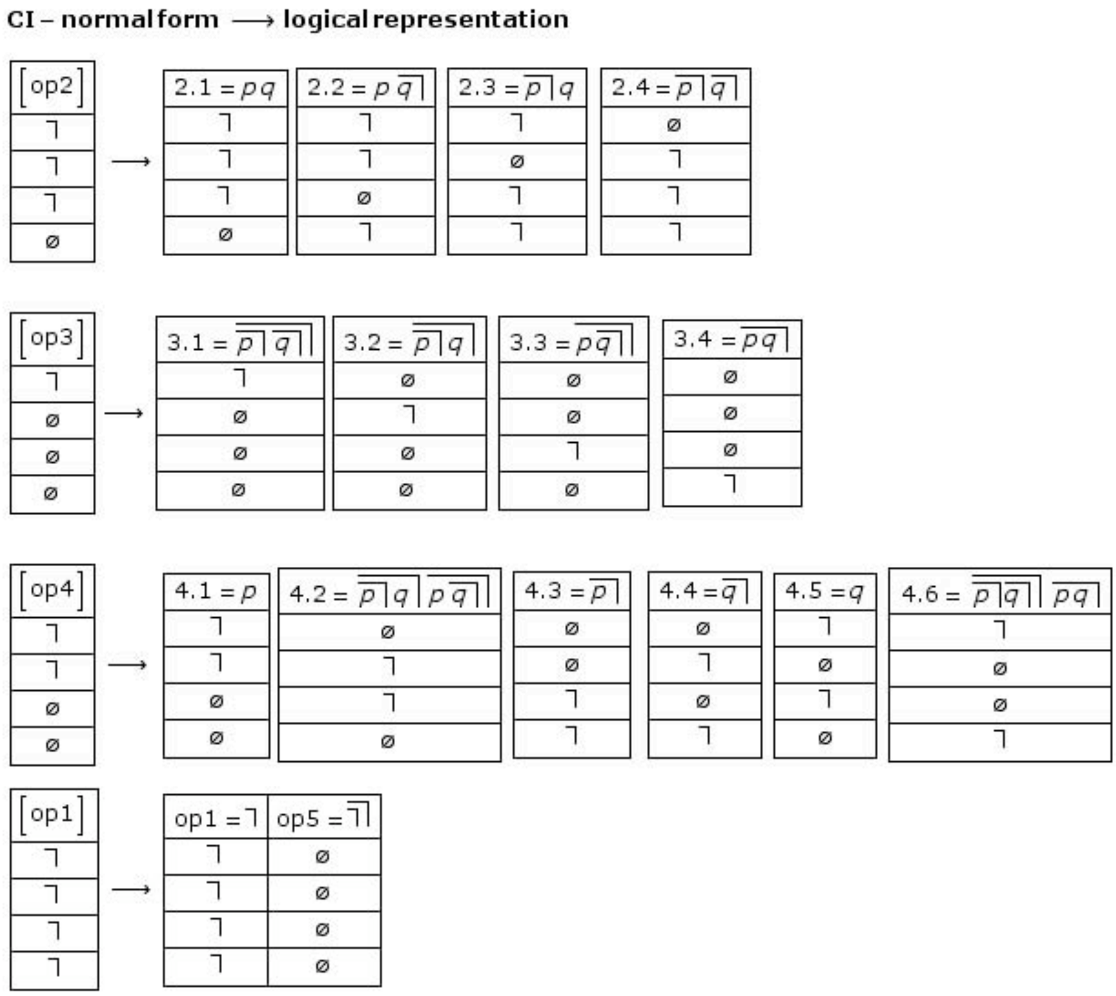
Hence, the CI-expression "$\overline{A \rceil B \rceil}\rceil$" represents 6 binary logical functions:

AB, BA
¬ (¬ A ∨ ¬ B), ¬ (¬ B ∨ ¬ A),
¬ (A ⊃ ¬ B), ¬ (B ⊃ ¬ A).

But that's not really the point. Because of the property of permutation-invariance of the basic elements, the constellations "op2" and "op3" are representing 4 different logical realization of the CI-expression. And the constellation "op4" is representing 5 logical functions. The constellations "op1" and "op5" are invariant. All together, the 5 CI-constellations are representing the full range of binary two-valued logical functions: 1+4+4+6+1 = 16.

This kind of modeling is not taking into account the commutativity of conjunction respectively disjunction as in GSB's model but is focusing on the primary structure of indicational forms. The commutativity might be added secondarily as a property of the logical connectives.

Because the DeMorgan laws and even Nicod's function or the Sheffer stroke are representable by the CI on the base of concatenation (in contrast to superposition), all the PC functions are representable too.



The listed constellations op1, op5, and op2 and op3, correspond to Kauffman's classification. The constellations op4 and op6 are not represented in Kauffman's graphs.
The second graph entails all constellations build by "binary" applications only, i.e. 4.2 = $\overline{p\rceil q}\,\overline{pq}$ and 4.6 = $\overline{p\rceil q}\,\overline{pq}$ are not considered.

The Brownian algebra is a system of 5 basic patterns, op1 - op5, and a concatenation/superposition operation. The calculus of indication is abstracted from this algebra and reduced to an *element*-oriented calculus of distinction, with the operator and element ⌉ with its supplement ∅, supported by the traditional concept of *variables* and equality, operated by concatenation on the base of linearized strings.

CI-Algebra = (Brown; op1, ..., op5, concatenation/superposition, ⌉, ∅, =).

Classification of the constellations
op1: "truth", tautology
op5: "untruth", contradiction
op2-op4: "contingency".

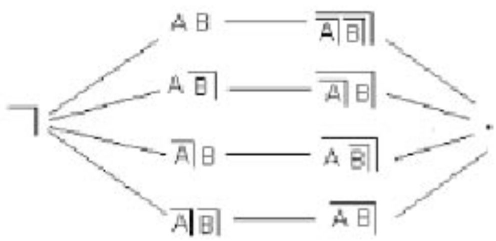
The CI is still a calculus of the dichotomy of "truth" and "untruth", with an emphasis on indicational "*truth*".

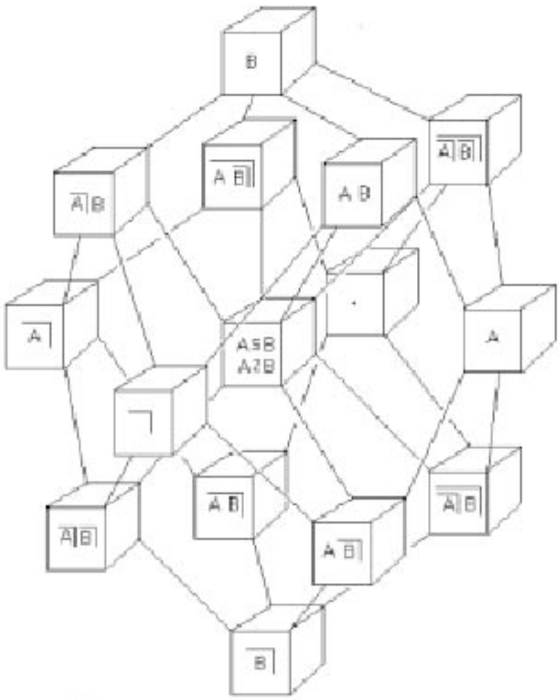
Consequences

Again, the relationship between CI-representations and CI-expressions has to be re-established.

[op1]: {op1, op5 } ∈ CI-rep ==> op1 $=_{CI}$ op5
[op2]: {op2.i, i=1, ..,5} ∈ CI-rep ==> op2.i $=_{CI}$ op2.j, ∀i, j=1, ...,5
[op3]: {op3.i, i=1, ..,4} ∈ CI-rep ==> op3.i $=_{CI}$ op3.j, ∀i, j=1, ...,4|
[op4]: {op4.i, i=1, ..,6} ∈ CI-rep ==> op4.i $=_{CI}$ op4.j, ∀i, j=1, ...,6

Kauffman’s graphs





The same again

BOOLEAN Name	Form	a -> ⏋ b -> ⏋	a -> ⏋ b -> ⏋⏋	a -> ⏋⏋ b -> ⏋	a -> ⏋⏋ b -> ⏋⏋
FALSE	⏋⏋	0	0	0	0
NOR	a b⏋⏋	0	0	0	1
NOT *a* AND *b*	a b⏋⏋⏋	0	0	1	0
NOT *a*	a b⏋⏋a b⏋⏋	0	0	1	1
a AND NOT *b*	a⏋b⏋⏋	0	1	0	0
NOT *b*	a⏋b⏋a b⏋⏋	0	1	0	1
XOR	a⏋b⏋a b⏋⏋⏋	0	1	1	0
NAND	a⏋b⏋a b⏋a b⏋⏋	0	1	1	1

(Michael Schneider)

Differentiational case

CD:

Mersenne: $\{a, b\}^2{}_{Mers} \longrightarrow \{a, b\}$:

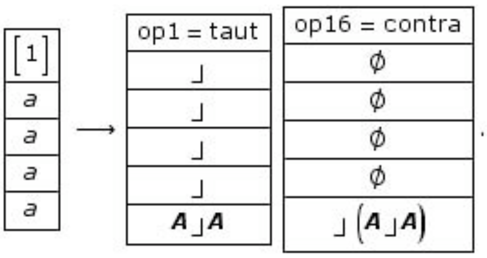
$M_n = 2^n - 1$, for $n = 2$: $2^4 - 1 = 15$.

Reduction quotient:

$$\text{reduction}\left(\frac{\text{Logic}(m, n)}{\text{Mersenne}(m, n)}\right) = \frac{m^{m^n}}{2^n - 1} : \frac{2^{2^2}}{2^4 - 1} = \frac{16}{15} = 1, 06.$$

Mersenne standard normal form

a	a	**b**	b	b	a	a	a	a	a	a	b	b	b	b
a	a	**b**	b	b	a	a	b	b	b	b	a	a	a	a
a	a	**b**	a	a	b	b	a	a	b	b	a	a	b	b
a	b	a	a	b	a	b	a	b	a	b	a	b	a	b
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15



Here, there are just two constellations that are coinciding:"op1" as logical tautology,"taut", and as logical contradiction, "contra". The concatenation is logically defined as a disjunction. The other constellations are accessible to a direct one-to-one logical interpretations.

Definition of logical functions by the interpretation of a set of Mersenne constellations.

A	*B*	┘*A*	┘*B*	┘*A*┘*B*	┘(┘*A*┘*B*)
┘	┘	∅	∅	∅	┘
┘	∅	∅	┘	┘	∅
∅	┘	┘	∅	┘	∅
∅	∅	┘	┘	∅	┘
op7	op9	op4	op14	op13	op10

AB	┘(*AB*)	┘*AB*	*A*┘*B*
∅	┘	┘	┘
┘	∅	∅	∅
┘	∅	∅	∅
∅	┘	┘	┘
op13	op10	op10	op10

Representation

{op1} ⟹ {op16} : 1 + 1

{op7, op9} ⟹ {op4, op10, op13, op14} : 4 + 2

{op7, op9, op2} ⟹ {op3, op5, op6, op8, op11, op12, op15} : 7 + 1

With the set {op1, op2, op7, op8} all 16 *logical* operations are represented.

op2	┘*A* op2	*A* op2	*B* op5	┘op2	Bop2	┘Bop2	┘*B* op5
┘	┘	∅	┘	∅	∅	┘	∅
┘	┘	∅	∅	∅	┘	∅	┘
┘	∅	┘	∅	∅	∅	┘	┘
∅	┘	∅	∅	┘	∅	┘	┘
–	op6	op5	op11	op3	op15	op8	op12

Because the necessary symmetry for the logical functions is not achievable with the Mersenne calculus, the corresponding generation of the PC functions out of a primary function on the base of concatenation, like Nicod or Sheffer, that is working for the CI, is not working for the CD.

Classification of the constellations
op1: "truth" = "untruth",
op2- op15: "contingency".
The CD is a calculus not so much of "truth" but of "*contingency*". Nevertheless it is still mainly focused on differentiational “truth".

Duality

op2 ≡ disj(A, B), op11 ≡ conj(A, B) :

op2 dual B(A(op2))

op6 dual op8 : ┘*A*(op2) dual ┘*B*(op2)

Comparison of unary constellations

Mersenne combinations

A	A⌋	⌋A	⌋A⌋	⌋A⌋	A⌋⌋	⌋A⌋⌋	A⌋⌋
⌋	⌋	∅	⌋	∅	∅	⌋	⌋
∅	⌋	⌋	∅	∅	∅	⌋	⌋

Mappings

Mersenne (p) : $\{\urcorner, \emptyset\}^2_{/CD} \longrightarrow \{\urcorner, \emptyset\}$: $2^2 - 1 = 3$

A⌋	A	⌋A
⌋	⌋	∅
⌋	∅	⌋

.

Short : CD = (a ≠ b, ab ≠ ba, aa = bb).

Contrast: Boolean m appings

Boolean (p) : $\{\urcorner, \emptyset\}^2 \longrightarrow \{\urcorner, \emptyset\}$: m^n **:** $2^2 = 4$

taut	A	¬ A	contra
⌝	⌝	∅	∅
⌝	∅	⌝	∅

.

Hom

X ∈ CD

⟹

val(X) = (⌋, ⌋),

val(⌋ X) = (∅, ∅)

⟹

X $=_{CD}$ ⌋ X

Example for CD

f = ⌋f : contradiction

f = ⌋ : ⌋⌋ = ∅

f = ∅ : ⌋∅ = ⌋.

Concatenation

$A = \lrcorner A \lrcorner = \lrcorner\lrcorner A$,

reversion $A = \lrcorner A$.

Equivalence for the calculus CD

$\text{hom}(X)$ iff $\forall x_i \in X : \text{val}(X) = \{\lrcorner\}$ or $\{\emptyset\}$,

$X \in \text{hom}(X) \Longrightarrow \underline{X}\lrcorner =_{CD} \lrcorner\underline{X}\lrcorner$: **CD – equivalence**

Constants

$\lrcorner = \underline{A}\lrcorner = \lrcorner\underline{A}\lrcorner\lrcorner = \underline{\underline{A}\lrcorner}\lrcorner$,

$\emptyset = \lrcorner\underline{A}\lrcorner = \underline{A}\lrcorner\lrcorner$.

Brownian combinations

A	$\overline{A}\urcorner$	$\urcorner A$	$\urcorner A \urcorner$	$\urcorner\overline{A}\urcorner$	$\overline{A}\urcorner\urcorner$	$\urcorner\overline{A}\urcorner\urcorner$	$\overline{\overline{A}\urcorner}\urcorner$	$\overline{\urcorner A}\urcorner$
$\urcorner$	$\emptyset$	$\urcorner$	$\urcorner$	$\urcorner$	$\urcorner$	$\urcorner$	$\urcorner$	$\emptyset$
$\emptyset$	$\urcorner$	$\urcorner$	$\urcorner$	$\urcorner$	$\urcorner$	$\urcorner$	$\emptyset$	$\emptyset$

Mappings

Brown(p): $\{\urcorner, \emptyset\}^2_{/CI} \longrightarrow \{\urcorner, \emptyset\}$: $\binom{n+m-1}{n}$: $\binom{2+2-1}{2} = 3$

$\urcorner A$	A	$\overline{\urcorner A}\urcorner$
$\urcorner$	$\urcorner$	$\emptyset$
$\urcorner$	$\emptyset$	$\emptyset$

.

Short : $CI = (a = a,\ a \neq b,\ ab = ba)$.

Example

$f = \overline{f}\urcorner$: equivalence; $A =_{CI} \overline{A}\urcorner$

$f = \urcorner : \overline{\urcorner}\urcorner = \emptyset$

$f = \emptyset : \overline{\emptyset}\urcorner = \urcorner$.

Superposition

$A = \overline{\overline{A}\urcorner}\urcorner$,

inversion$(A) = \overline{A}\urcorner$.

Constants

$\urcorner = \urcorner A = \urcorner A \urcorner = \urcorner\overline{A}\urcorner = \overline{A}\urcorner\urcorner = \urcorner\overline{A}\urcorner\urcorner$.

Equivalence for the calculus CI

$\text{perm}(X) \text{ iff } \forall x_i \in X : \text{val}(\text{perm}(X)) = \text{val}(X).$

$X \in \text{perm}(X) \Longrightarrow \text{perm}(X) =_{CI} (X)$: **CI - equivalence**

Example:

$X \in \text{perm}(X) : \overline{X}\rceil =_{CI} X.$

Het

$X \in CI$

$\Longrightarrow$

$\text{val}(X) = (\urcorner, \varnothing),$

$\text{val}(\overline{X}\rceil) = (\varnothing, \urcorner)$

$\Longrightarrow$

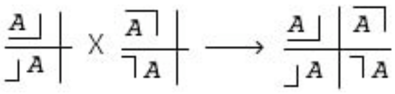
$X =_{CI} \overline{X}\rceil$

The *pattern-oriented* approach is taking the fact of the permutative abstraction for the CI directly into account. Therefore the initials or axioms have to be adjusted with "Het: $X =_{CI} \overline{X}\rceil$" to the new situation. (cf. *Appendix* 8.3.2)

This approach and result is obviously NOT in correspondence with the classical definition of the LoF.
It might be speculated that with this new understanding of the *Laws of Form*, the specific novelty of the CI is getting much more accessible than with the propagated classical approach.

The proposed CA-modeling of the behavior of the CI and the CD is adequately implemented with the rules of their cellular automata.

Complementary crossing



8.2.5. A further Stirling turn

As demonstrated in this paper, the differences between the calculus of indication and the calculus of differentiation becomes obvious with a complexity of just n=2 (and elements m=2). In the same sense logical basics like semantics are well defined. This is not anymore the case for Stirling calculi that are based on the trito-structure of graphematics. Here a complexity of at least n>=3 is necessary to demonstrate significant calculable differences between the Stirling constellations.

As it is obvious, the operators of distinctions and differentiations are very close to the combinatorial operations of permutation. The logical negation is clearly a permutation of logical values. Indication is changing the states of configurations from the marked to the unmarked and back. Differentiation has similar properties. This fact offers two different readings: an element- and a pattern-oriented understanding.

On the level of trito-structures, i.e. with the introduction of a Stirling calculus, the pattern property becomes dominant. Therefore, such trito-patterns are called *morphograms*, and their 'elements' *kenograms*. The position of a kenogram in a morphogram becomes crucial for the definition of Stirling patterns, i.e. Stirling morphograms. The identification of kenograms are depending on the position of the kenogram in a morphogrammatic pattern.

The operations of distinction and differentiation are generalized and applied as reflectors to morphograms. What played a relatively reduced role in the CI and the CD, *concatenation* and *superposition*, becomes a significant feature in the prominent roles as *coalitions* and *cooperations,* supported by other genuine morphogrammatic operations.

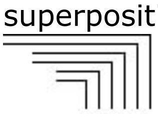
Like concatenations are "additions" and superpositions "multiplications" of strings, Stirling *coalitions* are replacing CI- and CD-concatenation, and Stirling *cooperations* are replacing CI- and CD-superpositions. Furthermore, the whole apparatus of deconstructed identity or equality relations has to be applied: *equality, equivalence, similarity, bisimilarity* and *metamorphosis*.

Superposition for the CI and the CD is not differentiated. It is realized by an unstructured monadic operator. In contrast, operators in Stirling systems are structured. A morphogram is not just an operand but an operator too. This fact is reflected by the structured coalitions and cooperations.

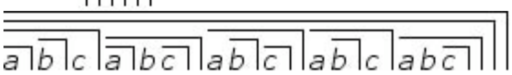
Because this figure of thought is definitively abandoning classical features dictated by identity thinking, the best exemplification, until now, is its interpretation in the framework of *morphic cellular automata*. A morphogram, then, is at once a *pattern* (morphé) and a *rule* (dynamis) of transclassical computation.

Despite the big ambitions declared by GSB with his calculus of indication as having surpassed the distinction of operator and operand, the manoeuvre works only for the highly trivial case of an unstructured operator, the cross. There is no chance, and obviously also no intention, to invent operators able of superposition of higher complexity, superpositions of structured distinctions. Obviously, something like "Q(P(X))" with Q and P defined as composed superposition terms is excluded from the world of forms. And certainly, such a case is not an example for a recursive form. The *Laws of Form* have a succession in the *Laws of Formations* (The Calculus of Idempositions, GSB, L. Kauffman, 1968) preserving the principal features of LoF.

What is possible for the CI is a highly restricted form of iteration as a superposition of the cross: From singular iteration:

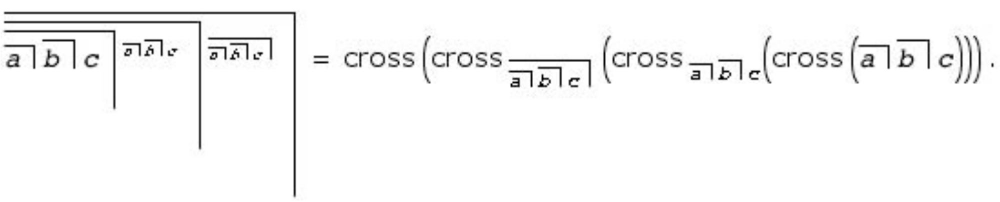


to composed formulas with iterations:



(Wolfram rule).

An example of structured superposition



The aim of GSB's *Laws of Form* is to give the most simple calculus of form. As shown, there is at least a complementary calculus of similar simplicity, the calculus of differentiation. Hence, the uniqueness of the CI simplicity is disturbed by its "mirror" image.

May be, simplicity is not anymore what we are looking for.

The cultural belief into the metaphysical doctrine: *"the emptiness is empty"* and the *"void is void"*, etc., has become as shallow as Edinburgh's famous Ghost House. Everybody who knows *Loch Ness* knows about the dynamics of emptiness, the Lochness of the Loch Ness.

8.2.6. A new look on contradictions with reentry forms

An interesting consequence, still to study, is the application of the new situation to *reentry* forms.
Take 12.2: $f = {}_{CI}\overline{f}\rceil$, then the usual construction of a contradiction, as shown by Varela, becomes questionable, again.
With the application of the initial $\overline{\overline{p}\rceil p}\rceil = \varnothing$ to the formula "$f = {}_{CI}\overline{f}\rceil$", we have, according to Varela's construction:
$f = {}_{CI}\overline{f}\rceil$, then
$= \overline{\overline{f}\rceil f}\rceil$
$= \overline{ff}\rceil$, with substitution
$= \overline{f}\rceil$
$= f$, with substitution.
"*But this leads to a contradiction, because substituting in (12.2) we have*
$= \rceil$, $[\varnothing = \rceil]$
which would render the calculus of indication inconsistent, and thus useless, by confusing every form." (Varela, p. 128).

This result is obviously based on the "*element-oriented*" approach that is excluding any pattern-oriented considerations.
A pattern-oriented understanding of the calculus would clearly deliver a different result. Following the sketched introduction to the pattern-oriented CI we would not get "$\varnothing = \rceil$", which is obviously a contradiction based on an atomic constellation, but the pattern "$\varnothing\varnothing = \rceil\varnothing$".

But the different approach is directly applicable to Varela's start formula "$f = {}_{CI}\overline{f}\rceil$".
For f = p we get val(p) = $(\rceil, \varnothing)$ and val($\overline{p}\rceil$) = $(\varnothing, \rceil)$ and applying Het, we get $p = _{CI}\overline{p}\rceil$, and therefore, by substitution, $f = {}_{CI}\overline{f}\rceil$, which holds in the pattern interpretation of the CI as an *equivalence relation*. With that, the Brownian emphasis on the difference between the CI and negational logic gets some further justification.

With that, the still identity-logic inspired construction Varela's is losing all its resonability.
It is also an open question if this direct modeling of the reentry form corresponds to the Brownian approach. To my knowledge, reentry occurs in the CI only in *context* with a form defined by other constituents, i.e. variables, and not in isolation.

A more profound analysis of the situation, considering polycontexturality, is given by Elena Esposito in *Ein zweiwertiger nicht-selbständiger Kalkül*:

"Wollte man dies formalisieren, müßte man auf einen Kalkül von Wiedereintritten rekurrieren, das heißt auf einen Kalkül von Beziehungen nicht mehr zwischen Bezeichnungen, sondern zwischen Beobachtungsordnungen. Das jedoch ist nicht mehr die Aufgabe des Indikationenkalküls." Elena Esposito, *Ein zweiwertiger nicht-selbständiger Kalkül*, in: Kalkül der Form, (Dirk Baecker, hrsg.), suhrkamp taschenbuch, Frankfurt/M. 1993, p.96-111.

8.2.7. Reflector analysis

Apart of a logical interpretation of CI-constellations, there is a reflector-oriented analysis of direct interest. The merits of a reflector-approach becomes evident in a polycontextural framework of complexions of CIs (Kaehr, 1981). The reflector-analysis was successfully applied for the study of the behavior of composed morphograms in morphogrammatics.

The calculus of indication intents to strip down the logical “corset” of variables and functions down to the structural bones of its mechanism. Therefore, a reflectional analysis is achieving more directly the attempts of a skeleton-analysis than the common approaches of logical comparisons. A sketch of a reflector-analysis is given for Brown, Mersenne, Stirling and Deutero-structures.

Reflector analysis for CI

$$\text{refl}\left(op_i\right) = op_i, \text{ for } i = 1, 4, 5 \quad : \text{self-reflective}$$
$$\text{refl}\left(op_2\right) = op_3, \quad : \text{reflective}$$
$$\text{refl}\left(\text{refl}\left(op_i\right)\right) = op_i, \ i = 1, \ldots, 5 \ : \text{self-reflective}$$

Reflector analysis for CD

refl(1) = (1) refl(4) = (7)
refl(2) = (12) refl(5) = (15)
refl(3) = (11) refl(6) = (9).
(1) : self-reflective
(i), i = 2,...,15 : reflective

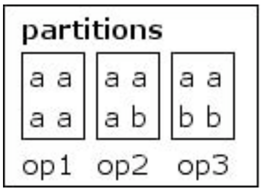
Reflector analysis for Stirling

Stirling

op1	op2	op3	op4	op5	op6	op7	op8
a a	a a	a a	a b	a a	a b	a b	a b
a a	a b	b a	a a	b b	a b	b a	b b

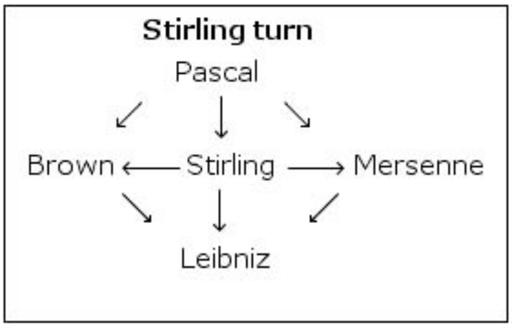
refl(1) = (1), refl(5) = 5, refl(6) = (6), refl(7) = (7) : self-reflective
refl(2) = (8) : reflective,
refl(3) = (4) : reflective.

Reflector analysis of deutero-structures



refl(i) = (i), i = 1, 2, 3.

This short analysis of different reflector-systems makes the position of Mersenne and Brownian calculi inside the graphematic system more clear.



8.3. Appendix 3: A second look at the calculi

8.3.1. Pattern oriented CI and CD

Conflicts with the primary intentions

Calculi shall be developed that are dealing not just with atomic elements (states) but with *patterns* of behaviors, as far as this is possible within the framework of calculi like the CI and the CD.

With the introduction of the CI- and the CD-**equivalence** relation over CI-expressions *e* and CD-expressions *d* the connection between the *intentions* of the calculi CI and CD as founded in their graphematical systems is re-established.
It seems, that there is a *discrepancy* in the formalization of the calculus of indication and its intention. This intention seems to be properly represented by the graphematics of the CI-intention.
Similar holds for the formalization of the calculus of differentiation CD.
The gap in the formalizations is produced by the mapping of expressions of sets of atomic "states" to sets of atomic "states", in the CI the states Cross and NoCross, i.e. from {⏋, ⌀} to { ⏋, ⌀} and in the CD from {⌋, ∅} to the "states" {⌋, ∅}.

Hence, the use of a *functional* representation, i.e. F:$\{x, y\}^n$ --> {x, y}, without an **abstraction** from the permutation of states for the CI, i.e. $F_{/perm}$, and the abstraction from the equality of homogeneous states for the CD, i.e. $CD_{/hom}$, seems to mislead the intentions of the interventions. The transition from the "*primary arithmetic*" to the "*primary algebra*" with its use of variables is based on the assumption of the correctness of the step-wise atomic "*substitution*", e.g. "Let p = ⏋. Then". This shows the unresolved complicity with "identity" assumptions, well in contrast with the claims of the *Laws of Form.*

Albeit differently motivated, and not touching the fundaments of the CI formalization at all, a similarity to the above remarks could be constructed with Varela's reflection axiom "a = $\overline{a}\rceil$" (12.27) and "p($\overline{a}\rceil$) = a" for his *"generalized Brownian algebra"* of "*periodic sequences*" and "*wave forms*"(Varela, p. 150/51). But Varela's "transposition algebra" of "wave forms" is not considered as a proper Brownian calculus for forms.

The aim of pattern-oriented formalization is defined by the attempt to take directly into account the graphematic structure of the CI and CD calculi as it appears within its tree and graph productions.

The fact that the expressions e_1= (aaba) and e_2= (aaab) of the primary algebra are indicationally equivalent is not properly represented by the ordinary formalization(s) of the *Laws of Form*.

The same holds for the CD. The fact that the expressions d_1= (aaaa) and d_2= (bbbb) are differentially equivalent is not properly represented by the proposed formalization of the *Laws of Differentiation*. (cf. Appendix 8.2.4)

CI:
e_1= (abaa)∈graph with the interpretation (⏋∅⏋⏋) and e_2= (aaab)∈graph with the interpretation (⏋⏋⏋∅) results in the conflictive equation: e_1 != $\overline{e_2}\rceil$.
From the point of view of the graph-generation for the CI, both patterns, $e_1$and e_2, are CI-equal.
Hence, the pattern related fact of *permutative* equality has to be considered: X $=_{CI}$ perm(X).

CD:
d_1= (aaaa)∈tree with the interpretation (⌋⌋⌋⌋) and d_2= (bbbb)∈tree with the interpretation (∅∅∅∅) results in the conflictive equation: d_1 != d_2.
From the point of view of the tree-generation for the CD, both patterns, $d_1$and d_2, are CD-equal, d_1 $=_{CD}$ ⌋d_1.
Hence, the pattern related fact of *homogeneous* equality has to be considered: X $=_{CD}$ hom(X).

8.3.2. Definition of a calculus of indication

A different "*calculus taken out of the calculus".*

Following the standard definitions of a calculus, the simple tectonics of alphabet, syntax and interpretation, "semantics", proof theory has to be introduced.

Calculus of Indication
Alphabet
constants = {⏋, ∅}
variables = {p, q, r, ...}

Syntax
concatenation of the constants {⏋, ∅} and superposition of {⏋}.
R0.1: ==> ⏋
R0.2: ==> ∅
R1.1 n ==> n ⏋
R1.2 n ==> $\overline{n}\rceil$
R1.3 n ==> n ∅

Concatenation and superposition of variables
R3.1: p ==> $\overline{p}\rceil$
R3.2: p, q ==> p q

With the rules R1 - R3 a production of strings based on the alphabet is defined. All the CI-strings produced are accepted as CI-expressions.

Example
e = ppq⌉ ⌉ q ∅ $\overline{p}\rceil$ r ∅ ⌉ $\overline{q \varnothing \overline{p}\rceil r \varnothing \rceil}\rceil$.

Hence, any expression build on the base of CI-constants and CI-variables with the operators of *concatenation* and *superposition* of the constant ⌉ is a CI-expression.

Semantics, i.e. indicational rules
J1: ⌉ ⌉ ⇔ ⌉
J2: $\overline{\rceil}\rceil$ ⇔ ∅

I1: $\overline{\overline{p}\rceil p}\rceil$ ⇔ ∅
I2: $\overline{\overline{pr}\rceil\ \overline{qr}\rceil}\rceil \leftrightharpoons \overline{\overline{p}\rceil\ \overline{q}\rceil}\rceil r$

Valuation
V1: The valuation of variables, p_1, p_2, ..., p_n is defined by the mapping val(CI): $\{\rceil, \varnothing\}^n$ --> {⌉, ∅}.

Wording for the valuation of variables:
"Let p = ⌉. Then ..."
"Let p = ∅. Then..."
This defines a valuation *table*, like for the semantics of propositional logic but without a semantic-ontological commitment.

P: *Rules for proofs*, based on substitution and equality.

With this setting, and its further elaboration, the crucial situations of the CI as they are determined by the graphematic graph-structure of the indicational events are *not* approached at all.

Therefore, the semantic mapping for the variables has to be specified according to the "*pattern*"-structure of the indicational graph. With that, the algebraic rules are not just depending on the atomic values (constants) but on the structure of the valuation too. There are just two classes of patterns: *homogenous* and *heterogenous* to consider at first.

Homogeneus patterns are treated like atomic constants, i.e. X ∈ Hom ==> X != $\overline{X}\rceil$.
Heterogeneous patterns are permutation-invariant, therefore: X ∈ Het ==> X = $\overline{X}\rceil$.

Therefore, according to the proposed conception of indication, the calculus of indication has to be specified by the meta-theoretic distinction of *homogenous* and *heterogeneous* valuations of its variables, i.e. X ∈ Hom or X ∈ Het.

Heterogenuos patterns are *classified* according the number of their elements.
The number of all patterns for an indicational graph are given by the

combinatirial formula:

$$val(CI) : \{\urcorner, \varnothing\}^{n}/_{Ind} \longrightarrow \{\urcorner, \varnothing\} \text{ with } Ind_{(n,m)} = \binom{n+m-1}{n}$$

Because there are only two homogeneous constellation in an indicational graph for m = 2, the number of heterogenous situations is:

$$Het(CI) = Ind_{(n,m)} - 2.$$

Classification Het – Hom

1. $X \in Het$ iff $val(perm(X)) \in CI : (X \in Het \Longrightarrow X =_{CI} \overline{X}\urcorner) \in CI$, or

$$val(X) = val(perm(X)) \Longrightarrow (X =_{CI} \overline{X}\urcorner) \in CI$$

$$\boxed{X \in Het \Longrightarrow X =_{CI} \overline{X}\urcorner}$$

2. $X \in CI : X \in Hom$ iff $perm(X) = X$:

$$\boxed{X \in Hom \Longrightarrow \overline{X}\urcorner \neq_{CI} X}$$

3. $\forall X : X \in CI \Longrightarrow X = \overline{\overline{X}\urcorner}\urcorner$.

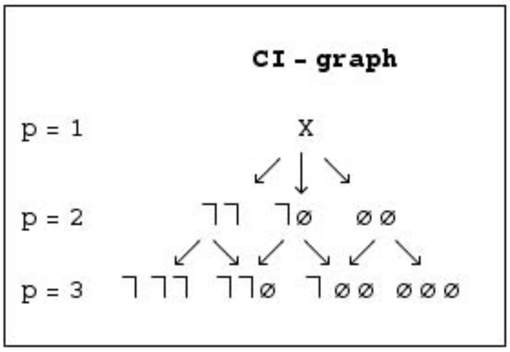
Short :

$$val(X) = val(perm(X)) \Longrightarrow X =_{CI} \overline{X}\urcorner,$$

$$val(X) = val(perm(X)) \Longrightarrow X =_{CI} \overline{perm(X)}\urcorner,$$

$$val(X) \neq val(perm(X)) \Longrightarrow X \neq_{CI} \overline{perm(X)}\urcorner.$$

Example



$p = 2.1: p = ⅂⅂ \Longrightarrow \overline{p}\rceil \neq p : \text{Hom}(p)$

$p = 2.2: p = ⅂\varnothing \Longrightarrow \overline{p}\rceil = p : \text{Het}(p)$

$p = 2.3: p = \varnothing\varnothing \Longrightarrow \overline{p}\rceil \neq p : \text{Hom}(p).$

Consequences for morphCI

The classical CI – Axiom J1, Position : $\overline{\overline{p}\rceil p}\rceil = \varnothing$, gets a reformulation in a morphic CI, morp

morphCI – Position

$p \in \text{morphCI} : \overline{\overline{p}\rceil p}\rceil = \overline{p}\rceil$

Explanation

$\overline{\begin{pmatrix} ⅂ \\ \varnothing \end{pmatrix}\rceil \begin{pmatrix} ⅂ \\ \varnothing \end{pmatrix}}\rceil = \overline{\begin{pmatrix} \varnothing \\ ⅂ \end{pmatrix} \begin{pmatrix} ⅂ \\ \varnothing \end{pmatrix}}\rceil = \overline{\begin{pmatrix} ⅂ \\ \varnothing \end{pmatrix}}\rceil = \begin{pmatrix} \varnothing \\ ⅂ \end{pmatrix} = \overline{p}\rceil$, hence with $p =_{CI} \overline{p}\rceil$:

$\overline{\overline{p}\rceil p}\rceil = \overline{p}\rceil.$

CI - equalities for two variables

$2.\ p\,q = p\,\overline{q}\rceil = \overline{p}\rceil\,q = \overline{p}\rceil\,\overline{q}\rceil \in \text{Hom2}$

$3.\ \overline{\overline{p}\rceil\,\overline{q}\rceil}\rceil = \overline{\overline{p}\rceil\,q}\rceil = \overline{p\,\overline{q}\rceil}\rceil = \overline{p\,q}\rceil \in \text{Hom3}$

$4.\ p = q = \overline{\overline{p}\rceil\,q}\rceil\overline{p\,\overline{q}\rceil}\rceil = \overline{p}\rceil = \overline{q}\rceil = \overline{\overline{p}\rceil\overline{q}\rceil}\rceil\ \overline{p\,q}\rceil \in \text{Hom4}.$

CI - inequalities

$1.\ ⅂ \neq \overline{⅂}\rceil,$

$5.\ p\,q \neq \overline{p\,q}\rceil \neq \overline{p}\rceil.$

8.3.3. Definition of a calculus of differentiation

Recalling the usual machinery for calculi, propositional logics and the sketch of a definition of the calculus of indication, the procedure to establish a calculus of differentiation follows the same strategies.

Here, this manoeuvre shall be reduced to the essentials. And that is the pattern aspect of the homogeneous constellations of “valuations".

Classification Het – Hom

1. $X \in \text{Het}$ iff $val(perm(X)) \in CD$: $(X \in \text{Het} \Longrightarrow X \neq_{CD} \rfloor X) \in CD$, or

$val(X) = val(perm(X)) \Longrightarrow (X \neq_{CD} \rfloor X) \in CD$:

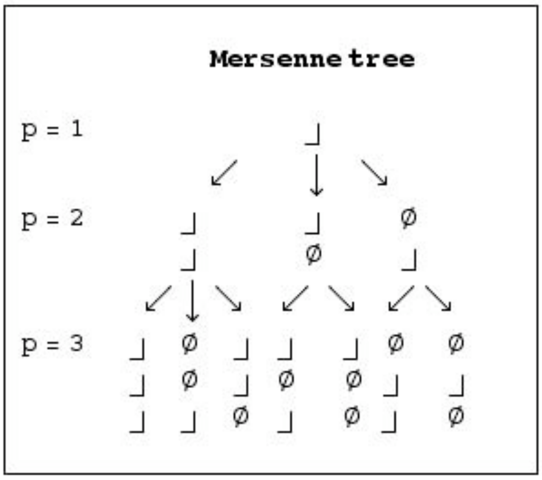
$$X \in \text{Het} \Longrightarrow \rfloor X \neq_{CD} X$$

2. $hom(X) \in CD$ iff $\forall x_i \in X : val(X) = \{\rfloor\}$ or $val(X) = \{\emptyset\}$:

$$X \in \text{Hom} \Longrightarrow \rfloor X =_{CD} X$$

.

Example



$p = 2.1 : p = \rfloor \rfloor \Longrightarrow \rfloor p = p$: $\text{Hom}(p)$

$p = 2.2 : p = \rfloor \emptyset \Longrightarrow \rfloor p \neq p$: $\text{Het}(p)$

$p = 2.3 : p = \emptyset \rfloor \Longrightarrow \rfloor p \neq p$: $\text{Het}(p)$.

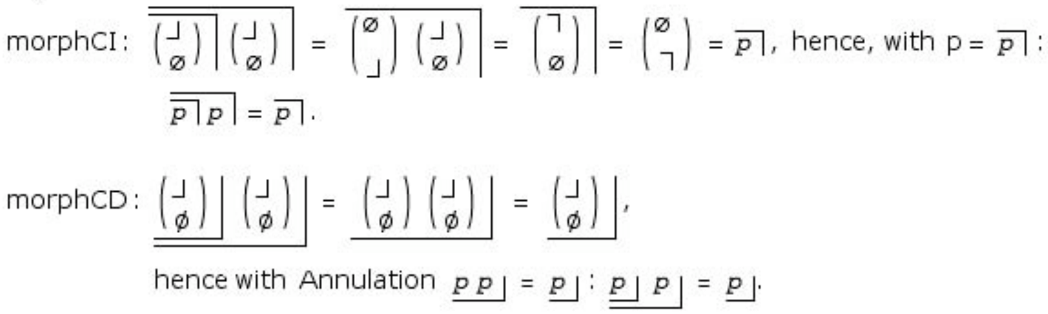
Consequences for morphCD

The classical CD – Axiom N1, Position : $\underline{p \rfloor\, p} \rfloor = \rfloor$, gets a reformulation in a morphic CD as :

morphCD – Position

$p \in \text{morphCD} : \underline{p \rfloor\, p} \rfloor = \underline{p \rfloor}$

Explanation



8.4. Appendix 4: Mersenne analysis of Brownian de/inscriptions

Second-order formulas have different motivations in Spencer-Brown's Laws of Form. A new turn is achieved with the introduction of the complementary Mersenne calculus. The CI is studying distinction in or of a world, while the CD is studying the differentiations of inscriptions of calculi in a graphematic (scriptural) space. Hence, a new second-order type of analysis is opened up: the differentiational study of distinctional inscriptions, i.e. the CD-study of CI-notations.

$$X \in \mathrm{CI} \Longrightarrow XX = X \in \mathrm{CI}\,. \Longrightarrow .\, XX = \emptyset \in \mathrm{CD}$$

This is similar to GSB's the study of Russell/Whitehead's logic with the means of his CI. This analysis of logic by the CI offers a specific reduction of the logical material: *"this represents a reduction of the mathematical noise-level by a factor of more than 40000"* (LoF, 1972, p. 117)

Hence, a minimal graphematic *comparistics* of formal languages is in place: logic, distinctional and differentional calculi.

A self-application of CI formulae are a reentry of the form into the form.
A CD self-application is analyzing the form of the CI.

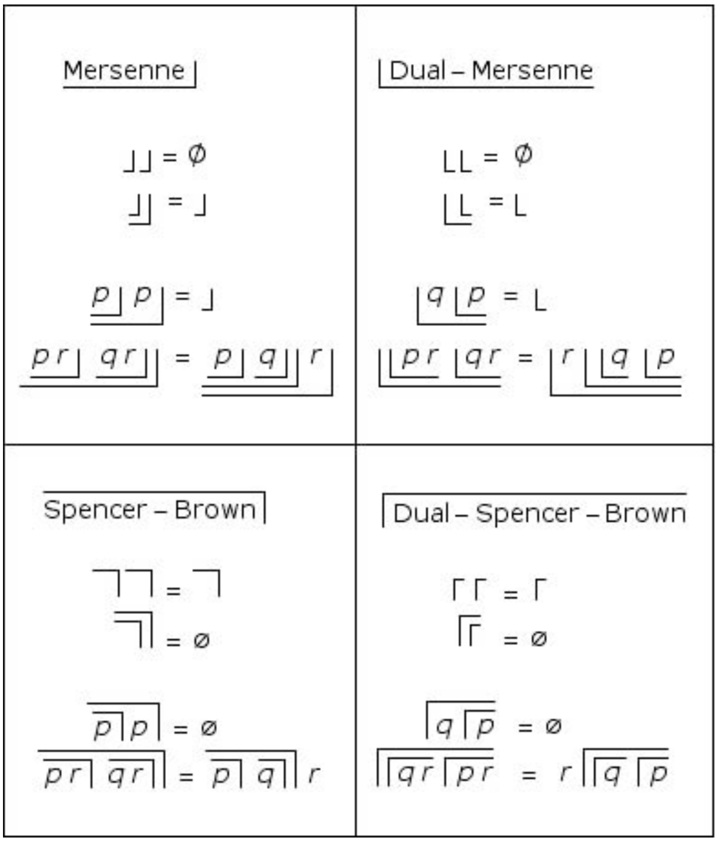
8.5. Appendix 5: Quadralectics of distinction and differentiation

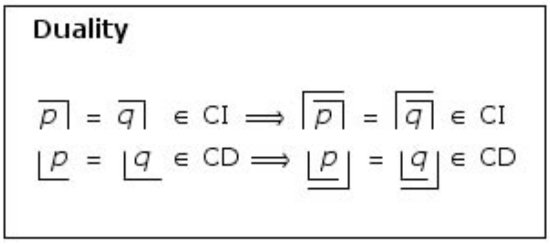
As we learned from the papers "*Diamond calculus*" and "*Quadralectics*", there is a precise architecture and mechanism of 4-fold disseminations of calculi, esp. the calculus of indication. What wasn't established in the previous papers was an exact specification of the laws of a "*complementary*" calculus of indication, i.e. the Mersenne calculus. Dual-forms to existing calculi had been well defined but not elaborated.

With the approach to Mersenne calculi the promise for a complementary calculus is fulfilled and the gap is finally closed.

Quadralectics of CI and CD

$$\mathrm{Quadraletics}^{(2,2)}\ (\mathrm{CI}, \mathrm{CD}, \mathrm{CI}^{D}, \mathrm{CD}^{D}) = \begin{array}{c|c} \text{Mersenne} & \text{Mersenne}^{D} \\ \hline \text{Spencer-Brown} & \text{Spencer-Brown}^{D} \end{array}$$





Complementary and inverse forms
Because of the principle of "*perfect continence*", there are no dual forms in a logical sense in the calculi of forms. What is reflecting the formation of forms are parallax and complementary, i.e. diamond formations of form. This is mirrored firstly, by the systems of inverse forms. Hence, the basic, and not yet disseminated planar forms, are the forms of *complementarity* and *inversion* (here: duality).

Complementary calculi are based on complementary *graphematic* systems. Therefore, there is no mechanism given by a calculus to define its complementary calculus by the means of the calculus. Dual systems remain in their graphematic framework, i.e. the dual of a calculus is defined by the means of the calculus itself. Hence, the graphematics of the CI is complementary to the graphematics of the CD, while the graphematics of the dual systems are unchanged.

Distinctions between distinction systems
Beyond the systematics of planar distinctions, a polycontextural theory and calculus of distinctions, is demanding for distinctions between discontextural distinction systems. This might be realized by the introduction of topological and

knot-theoretic constellations of distinction systems. A simple start could be a 3-dimensional distinction system with the set of planar distinctions and reentries at each contextural position and the transcontextural distinctions and reentry forms between distributed contextural distinction systems.

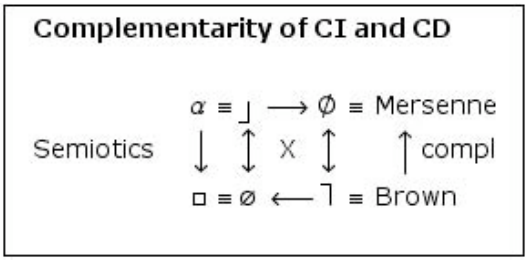
8.6. Appendix 6: Complementarity: A case for Diamond Theory

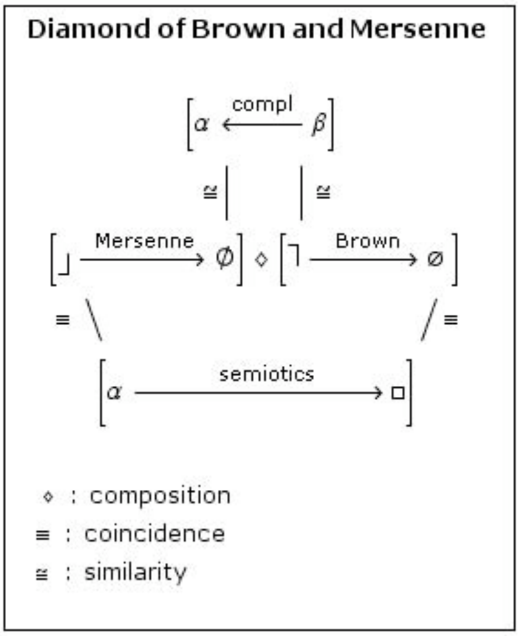
One of the main terms used in this presentation of a complementarity between the Spencer-Brown calculus of indication and the Mersenne calculus of differentiation is the term "*complementarity*" itself.
As shown in other papers, the most fundamental and most direct approach to define the term "complementarity" is, as far as I know, accessible by the theory of diamond category. Diamond category is defined as an interplay of categories and saltatories. Such an interplay defines the complementarity of categories and saltatories in diamonds.

One crucial motivation to use diamond categorical constructions is the fact that the "concatenation" of marks in the CI is not defined by properties of the calculus but based on cultural intuition. This lack of definition is fixed with the strategy of the matching conditions for morphisms.

Complementarity = diamond (Brown, Mersenne).

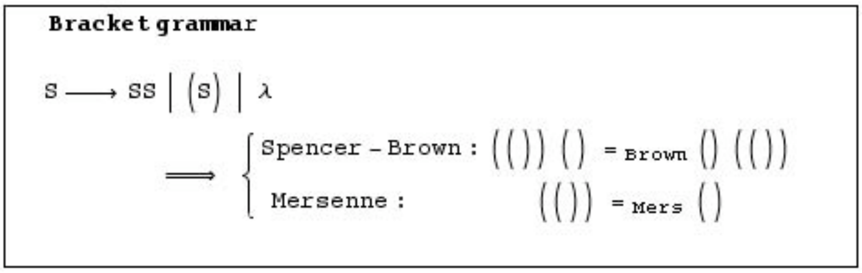
diamond (Mersenne ∐ Brown) = Mersenne || Brown

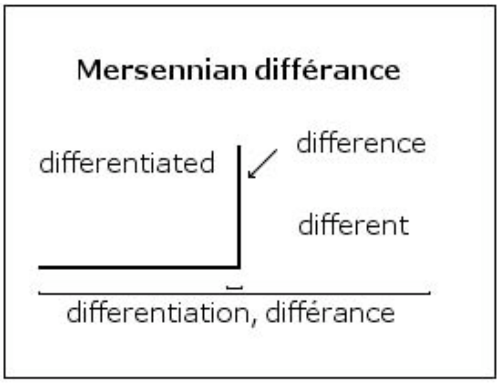


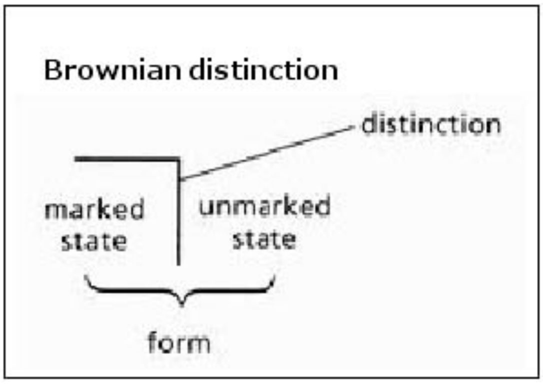


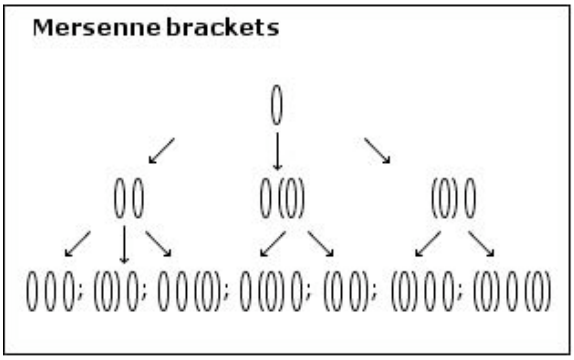
Semiotics in this chiastic diamond diagram is reflecting the ‘common’ and *acceptional* features of Brown and Mersenne, while complementarity is the place where the *rejectional* difference between both are kept together.
Given this chiastic diamond modeling, iterative and accretive *combinations* to build networks, grids or meshes out of it are following naturally.

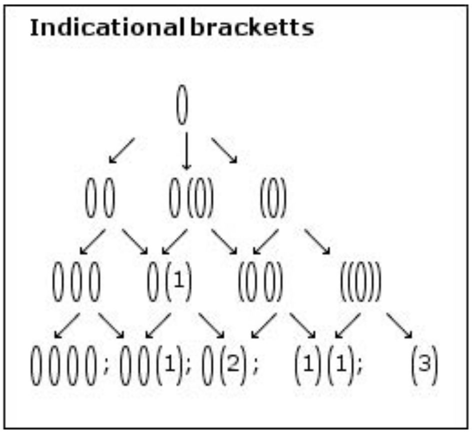
9. Index of Forms and Differentiations











Numeric indicational rules

R1: $\Longrightarrow \{1^1\}$

R2: $\{1^1\} \Longrightarrow \{1^2\} \mid \{1^1\,2^1\} \mid \{2^2\}$

R3.1: $\{1^n\} \Longrightarrow \{1^{n+1}\} \mid \{1^n\,2^1\}$

R3.2: $\{2^n\} \Longrightarrow \{2^{n+1}\} \mid \{1^1\,2^n\}$

R3.3: $\{1^n\,2^n\} \Longrightarrow \{1^{n+1}\,2^n\} \mid \{1^n\,2^{n+1}\}$

Numeric Mersenne rules

R1: $\Longrightarrow \{1^1\}$

R2.1: $\{1^n\} \Longrightarrow \{1^{n+1}\} \mid \{2^n\,1^1\} \mid \{1^n\,2^1\}$

R2.2: $\{1^n\,2^n\} \Longrightarrow \{1^n\,2^n\,1^1\} \mid \{1^n\,2^{n+1}\}$

R2.3: $\{2^n\,1^n\} \Longrightarrow \{2^n\,1^{n+1}\} \mid \{2^n\,1^n\,2^1\}$

CI – Meta	Spencer – Brown CI	Mersenne CD	CD – Meta
Arithmetics Condensation Cancellation Order, I1 Number, I2	$\rceil\rceil = \rceil$ $\overline{\rceil}\rceil = \varnothing$	$\lrcorner\lrcorner = \emptyset$ $\underline{\lrcorner}\rfloor = \lrcorner$	Number, M1 Order, M2
Algebra Position, J1	$\overline{\overline{p}\rceil p}\rceil = \varnothing$	$\underline{\underline{p}\rfloor p}\rfloor = \lrcorner$	Contra – Position N1
Transposition, J2	$\overline{\overline{pr}\rceil\,\overline{qr}\rceil}\rceil = \overline{\overline{p}\rceil\,\overline{q}\rceil}\rceil r$	$\underline{\underline{pr}\rfloor\,\underline{qr}\rfloor}\rfloor = \underline{\underline{\underline{p}\rfloor\,\underline{q}\rfloor}\rfloor r}\rfloor$	Contra – Transposition N2
Meta – Theory Duality	$p = q \longrightarrow \overline{p}\rceil = \overline{p}\rceil$	$p = q \longrightarrow \lrcorner p = \lrcorner q$	Duality
Simplification Completenes Theorem 11.9	uniqueness provable $e_1 \rceil e_2 = \rceil$	uniqueness provable $\underline{e_1\,e_2}\rfloor = \lrcorner$	Simplification Completeness
Heterogeneous Homogeneous constellations	$X \in Het \Longleftrightarrow X = X$ $X \in Hom \Longleftrightarrow X = \rceil X$ $X \in Het \Longrightarrow \overline{X}\rceil =_{CI} X$ $X \in Hom \Longrightarrow \overline{X}\rceil \neq_{CI} X$	$X \in Het \Longleftrightarrow X = X$ $X \in Hom \Longleftrightarrow X = \lrcorner X$ $X \in Het \Longrightarrow \lrcorner X \neq_{CD} X$ $X \in Hom \Longrightarrow \lrcorner X =_{CD} X$	Classification of constellations as Hom and Het

CI – Theorems	Spencer – Brown CI	Mersenne CD	CD – Th
Reflexion C1	p⌉⌉ = p	p⌋⌋ = ⌋ p⌋ = ⌋	Inflexion
Generation C2	pq⌉q = p⌉q	pq⌋q = p⌋q	◻
Integration C3	⌉p = ⌉	⌋p⌋ = ⌋ ⌋⌋p = ⌋p⌋ = p	◻
Occultation C4 I1 Varela	p⌉q⌉p = p	p⌋q⌋p⌋ = p⌋	◻
Iteration C5 (Confirmation)	p p = p	p p = ∅ p p⌋ = ⌋ p p⌋ = p⌋	Annulation
Echelon Prop 11.22	p⌉q⌉r⌉ = pr⌉ q⌉r⌉	p⌋q⌋r⌋ = pr⌋ q⌋r⌋⌋	Cascade
(aa) ≠ (bb)	p p⌉ = qq	p q⌋ = q⌋p⌋	(ab) ≠ (ba)
◻		p⌋p⌋p⌋ = p⌋ = ⌋ p p p = p	Iteration
Prop 11.21	p⌉q⌉⌉ p⌉q⌉ = p	p⌋q⌋⌋ p⌋q⌋⌋ = p⌋	◻
Prop 11.23 modification	p⌉qr⌉sr⌉⌉ = p⌉q⌉s⌉ p⌉r⌉	p⌋qr⌋sr⌋⌋⌋ = p⌋q⌋s⌋⌋ p⌋r⌋⌋⌋	◻

f = f⌉ ∉ CI f = ⌉ : ⌉ = ⌉⌉ = ∅ f = ∅ : ∅ = ∅⌉ = ⌉ ∈ ⊤ "never true nor false" *oscillation* f = …a⌉b⌉a⌉b⌉ f = fa⌉b⌉	f = f⌋ ∈ CD, Inflexion f = ⌋ : ⌋ = ⌋⌋ = ⌋ ∈ ID f = ∅ : ∅ = ∅⌋ = ⌋ ∉ ID "always true" *fixation*	permutation f = f⌋f Self – Quotation ⌊f⌋ ≡ **f** = **f**⌋**f** Self – reference **SR**: ⌊f⌋ ≡ **f** = ⌋**f** f = …a⌋b⌋a⌋b f = fa⌋b

Mersenne Mappings

Mersenne(p): $\{⌉, ∅\}^2_{/CD} \longrightarrow \{⌉, ∅\}$: $2^2 - 1 = 3$

A⌋	A	⌋A
⌋	⌋	∅
⌋	∅	⌋

Hom
$X \in \mathrm{CD}$
$\Longrightarrow$
$\mathrm{val}(X) = (\lrcorner, \lrcorner)$,
$\mathrm{val}(\lrcorner X) = (\emptyset, \emptyset)$
$\Longrightarrow$
$X =_{\mathrm{CD}} \lrcorner X$

Concatenation
$A = \lrcorner A \lrcorner = \lrcorner\lrcorner A$,
reversion $A = \lrcorner A$.
Equivalence for the calculus CD
$\mathrm{hom}(X)$ iff $\forall x_i \in X : \mathrm{val}(X) = \{\lrcorner\}$ or $\{\emptyset\}$,
$X \in \mathrm{hom}(X) \Longrightarrow \underline{X\lrcorner} =_{\mathrm{CD}} \lrcorner\underline{X\lrcorner}$: **CD – equivalence**
Constants
$\lrcorner = \underline{A\lrcorner} = \lrcorner\underline{A\lrcorner}\lrcorner = \underline{\underline{A\lrcorner}\lrcorner}$,
$\emptyset = \lrcorner\underline{A\lrcorner} = \underline{A\lrcorner}\lrcorner$.

Brownian Mappings
Brown(p): $\{\urcorner, \varnothing\}^2_{/\mathrm{CI}} \longrightarrow \{\urcorner, \varnothing\}$: $\binom{n+m-1}{n}$: $\binom{2+2-1}{2} = 3$

$\urcorner A$	A	$\overline{\urcorner A \urcorner}$
$\urcorner$	$\urcorner$	$\varnothing$
$\urcorner$	$\varnothing$	$\varnothing$

.

Superposition
$A = \overline{\overline{A\urcorner}\urcorner}$,
$\mathrm{inversion}(A) = \overline{A\urcorner}$.
Constants
$\urcorner = \urcorner A = \urcorner A \urcorner = \urcorner\overline{A\urcorner} = \overline{A\urcorner}\urcorner = \urcorner\overline{A\urcorner}\urcorner$.
Equivalence for the calculus CI
$\mathrm{perm}(X)$ iff $\forall x_i \in X : \mathrm{val}(\mathrm{perm}(X)) = \mathrm{val}(X)$.
$X \in \mathrm{perm}(X) \Longrightarrow \mathrm{perm}(X) =_{\mathrm{CI}}(X)$.

Het
$X \in \mathrm{CI}$
$\Longrightarrow$
$\mathrm{val}(X) = (\urcorner, \varnothing)$,
$\mathrm{val}(\overline{X\urcorner}) = (\varnothing, \urcorner)$
$\Longrightarrow$
$X =_{\mathrm{CI}} \overline{X\urcorner}$

Contrast : Boolean mappings

Boolean(p): $\{\urcorner, \varnothing\}^2 \longrightarrow \{\urcorner, \varnothing\}$: m^n : $2^2 = 4$

taut	A	$\neg A$	contra
$\lrcorner$	$\lrcorner$	$\emptyset$	$\emptyset$
$\lrcorner$	$\emptyset$	$\lrcorner$	$\emptyset$

.

Classification Het – Hom for CI

1. $X \in$ Het iff *val* (*perm* $(X)) \in$ CI : $(X \in \text{Het} \Longrightarrow X =_{\text{CI}} \overline{X}\rceil) \in$ CI, or
   val(X) = *val* (*perm* $(X)) \Longrightarrow (X =_{\text{CI}} \overline{X}\rceil) \in$ CI

$$X \in \text{Het} \Longrightarrow X =_{\text{CI}} \overline{X}\rceil$$

2. $X \in$ CI : $X \in$ Hom iff *perm* $(X) = X$:

$$X \in \text{Hom} \Longrightarrow \overline{X}\rceil \neq_{\text{CI}} X$$

CI – equalities for two variables

2. $pq = p\overline{q}\rceil = \overline{p}\rceil q = \overline{p}\rceil\overline{q}\rceil \in$ Hom2
3. $\overline{\overline{p}\rceil\overline{q}\rceil}\rceil = \overline{\overline{p}\rceil q}\rceil = \overline{p\overline{q}\rceil}\rceil = \overline{pq}\rceil \in$ Hom3
4. $p = q = \overline{\overline{p}\rceil q}\rceil\overline{p\overline{q}\rceil}\rceil = \overline{p}\rceil = \overline{q}\rceil = \overline{\overline{p}\rceil\overline{q}\rceil}\rceil\,\overline{pq}\rceil \in$ Hom4.

CI – inequalities

1. $\urcorner \neq \overline{\urcorner}\rceil$,
5. $pq \neq \overline{pq}\rceil \neq \overline{p}\rceil$.

Classification Het – Hom for CD

1. $X \in$ Het iff *val* (*perm* $(X)) \in$ CD : $(X \in \text{Het} \Longrightarrow X \neq_{\text{CD}} \lrcorner X) \in$ CD, or
   val(X) = *val* (*perm* $(X)) \Longrightarrow (X \neq_{\text{CD}} \lrcorner X) \in$ CD :

$$X \in \text{Het} \Longrightarrow \lrcorner X \neq_{\text{CD}} X$$

2. hom$(X) \in$ CD iff $\forall x_i \in X$: val$(X) = \{\lrcorner\}$ or val$(X) = \{\emptyset\}$:

$$X \in \text{Hom} \Longrightarrow \lrcorner X =_{\text{CD}} X$$

BIFUNCTORIALITY OF MERSENNE, BROWN and SEMIOTICS

$$\left(\mathcal{U}_1 \bigcap_{1.2} \mathcal{U}_2\right) \bigcap_{1.2.3} \mathcal{U}_3 = \emptyset$$

$$\mathcal{U}^{(3)} = \left(\mathcal{U}_1 \amalg_{1.2} \mathcal{U}_2\right) \amalg_{1.2.3} \mathcal{U}_3$$

$$\mathcal{U}_1 = \{\text{Brown}_2,\ \text{distinction}_2\}$$

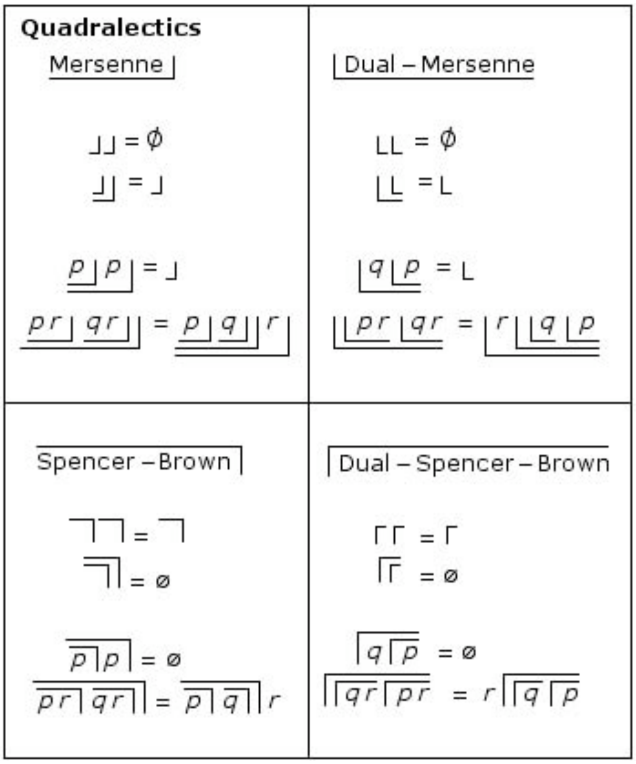
$$\mathcal{U}_2 = \{\text{Mersenne}_1,\ \text{differentiation}_1\}$$

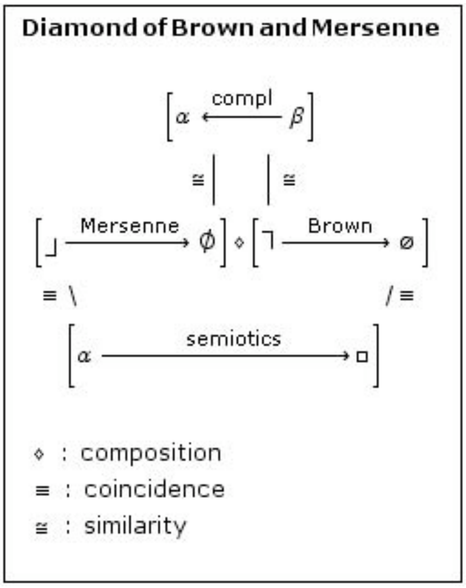
$$\mathcal{U}_3 = \{\text{Semiotics}_3,\ \text{concatenation}_3\} :$$

$$\begin{bmatrix} g_{Mers} & - & g_{Sem} \\ f_{Mers} & g_{Brown} & - \\ - & f_{Brown} & f_{Sem} \end{bmatrix} :$$

$$\left(\begin{matrix} \left(\begin{matrix} (f_{Mers} \circ^{1.0.0} g_{Mers}) \\ \amalg_{1.2.0} \\ (f_{Brown} \circ^{0.2.0} g_{Brown}) \end{matrix}\right) \\ \amalg_{1.2.3} \\ (f_{Sem} \circ^{0.0.3} g_{Sem}) \end{matrix}\right) = \left(\begin{matrix} \left(\begin{matrix} f_{Mers} \\ \amalg_{1.2.0} \\ f_{Brown} \end{matrix}\right) \\ \amalg_{1.2.3} \\ f_{Sem} \end{matrix}\right) \circ^{1.2.3} \left(\begin{matrix} \left(\begin{matrix} g_{Mers} \\ \amalg_{1.2.0} \\ g_{Brown} \end{matrix}\right) \\ \amalg_{1.2.3} \\ g_{Sem} \end{matrix}\right)$$

$\amalg$: mediation between contextures
$\circ$: composition of morphisms
$=$: equivalence





Structured versus unstructured superpositions

Unstructured

Structured superposition