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Promoting awareness for a not yet classified crime

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The Established Joy of Mental Abuse

Promoting awareness for a not yet classified crime

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Abstract

The aim is to promote awareness for a not yet classified crime of mental abuse.
The idea that there are different kinds of rationalities appears slowly to be recognized even in the pre-schooling context of Kindergartens. If there are different paradigms of rationality, children with deviating intellectual behaviors have to be accepted and it is the duty of a teacher to find out what kind of deviation the child is developing. Deviation is not necessarily a symptom of mental disorder. It just might signalize a mismatch between the child's unconscious and the teachers conscious paradigm of math.
The Leibniz approach is focusing on the identity of objects. The Brownian on the possibilities of partitioning a set of elements. The Mersennian on the possibilities of differentiations of the objects of a multi-set. While the Stirlingian deals with differences of elements of a pattern.
All four approaches are forcing different kinds of arithmetical and logical thinking.
(work in progress, vers. 0.3.5, Oct. 2013)

1. Kindergarten math: numbers and counting

1.1. How it starts

What are the aims of a standard western Kindergarten education in such abstract disciplines like math, geometry and counting?

The answer is easy found. Simply check the offers of one of the many educational organizations and supporting industries.

They all guarantee the parents a steep learning curve for their children to learn to master the basics of the adult mind set of math.

There is not a single offer that is taking the capacities of children seriously and offers strategies to develop genuine infant-adequate education.

One of the many successful companies is the company *"Home Schooling for Kids"* which offers "KS3, A-Levels, GCSE & IGCSE Courses From £350".

http://www.OxfordHomeSchooling.co.uk

What's on offer on the 'Sure Start' market?

"The goal of kindergarten math curriculum is to prepare children for first grade math. Please see below a list of objectives and goals for kindergarten math:

To count by rote at least to 20, but preferably a little beyond.
The concepts of equality, more, and less.
To count backwards from 10 to 0.
To recognize numbers.
To be able to write numbers.
To recognize basic shapes.
To understand up, down, under, near, on the side, etc. (basic directions).
To have a very basic idea of addition and subtraction.
It also helps to expose the student to two-digit numbers.

"Children may also get started with money, time, and measuring, though it is not absolutely necessary to master any of that. The teacher should keep it playful, supply measuring cups, scales, clocks, and coins to have around, and answer questions."

http://www.homeschoolmath.net/teaching/kindergarten.php

It is also important to know that the definition of a rational human being is implying the skills of those math topics added with the ability to draw some logical conclusions, say with *modus ponens*.

All that is instructed in the social context of a governmental schooling program that is confusing learning and training with education.

Failing such skills of adult cognition excludes the person to be qualified as a rational human being (homo sapiens).

Without surprise there is some resistance to the schooling movement.

"I suppose it is because nearly all children go to school nowadays, and have things arranged for them, that they seem so forlornly unable to produce their own ideas." Agatha Christie

http://studentliberation.com/quotes_1.html

In this paper, I will not deal with the many approaches of the anti-schooling movements but with the very essentials of conceptual thinking that are accepted by both sides, the schooling and the anti-schooling institutions and movements.

1.2. Some background theories for the traditional approach

Principles postulated in the tradition of the Piaget school

The abstraction principle

"The realization of what is counted is reflected in this principle. A child should realize that counting could be applied to heterogeneous items like toys of different kinds, color, or shape and demonstrate skills of counting even actions or sounds! There are indications that many 2 or 3 year olds can count mixed sets of objects.

The order-irrelevance principle

"The child has to learn that the order of enumeration (from left to write or right to left) is irrelevant. Consistent use of this principle does not seem to emerge until 4 or 5 years of age.

Constructivism approach

"There is strong evidence that the early teaching of standard procedures for arithmetic problem solving ***"thoroughly distorts in children's mind the fact that mathematics is primarily reasoning."*** *In order to address the above problem, new mathematics curricula have been introduced, based on the* ***Piaget theory of Constructivism.***

"This approach suggests that logico-mathematical knowledge, apart from empirical or social knowledge, is a kind of knowledge that each child must ***create*** *from within, in interaction with the environment, rather than acquire it directly (almost "being donated") from the environment."*

Natalia Marmasse, Aggelos Bletsas, Stefan Marti, *Numerical Mechanisms and Children's Concept of Numbers*
http://web.media.mit.edu/~stefanm/society/som_final_natalia_aggelos_stefan.pdf

These principles are subordinated under the binary question: *"To what extent is the sense of numbers* innate*, and to what extent is it* learned*?"* Nature or nuture?

"I learned most, not from those who taught me but from those who talked with me." St. Augustine

Again, we are told by Michael Gove the governments adviser: "*Genes make you smart, not teaching. Genetics outweighs teaching, Gove adviser tells his boss."*

"The most influential adviser to Education Secretary Michael Gove has penned a report in which he states that a child's genetics are more important than the teaching they receive."

http://www.theguardian.com/politics/2013/oct/11/genetics-teaching-gove-adviser

Without doubt, this poor guy has never studied the miserable arithmetics of modern genetics and DNA research.

There are good reasons to see the decision for a *dialog* with children as neither belonging to the "*nature*" nor to the "*nuture*" camp of the ongoing debate and battle. Dialogs are not in-forming children with educational content but are *evoking* their own yet hidden and developing capacities of thinking and understanding their own world.

Dialogical 'evocation' is not the technique to elicit neuro-morphic knowledge out of the little brain. It is a process that is not excluding surprises. Neuro-teaching is the training of concepts that have never been found in the brain.

It will turn out that exactly the postulated principles, the principle of *abstraction* and its corresponding principle of *order-irrelevance,* has no 'natural' foundation in the thinking process of a curious not yet educationally manipulated child. Nor are there any genetical conditions that are forcing to a specific kind of thinking numbers and logic.

Also children are taught the principle of "*order-irrelevance*" of counting, they are forced to write their results down from the left to the right.

As a young pupil I preferred to write from the right the left, and everything went well, until my teacher discovered the scandal. Then I was forced to change the direction of my early writing. Nobody was able to tell me the reasons and the advantage of the "correct" left-to-wright writing.

And surely, nobody wanted to understand how I succeeded to write the wrong way round in a social environment that behaved the opposite way round. Obviously, I was able to read and understand their writing.

Unfortunately the class was mono-ethnic and nobody from another culture could tell me that their grandparents are still writing successfully the other way round. At least I succeed to avoid a special treatment at a special school.

But this is just the 'iceberg' of the established narrow-mindedness.

It goes on with math. Successor and addition operations, e.g., are adding the units to the sequence of signs, shapes or numbers in one and only one direction. Why not 'backwards' and why not both, 'backwards' and 'forwards' together?

It might be the same for a math teacher, but for a child it makes a crucial difference.

I vaguely remember that there are crucial empirical results from genetics and brain research that are supporting this ideological decision.

Even if there would be strong empirical evidence and verification of a close connection between the concept of number and the genetical prepositions of the human brain it wouldn't stop the human mind to surpass such a little handicap.

Up to now we haven't detected any human beings that are able to study the moon with a naked eye nor do we know any genetically privileged children flying around the village without a little helicopter. And, certainly, the whole calculations for the scientific thesis wouldn't have been realized by non-assisted human brains alone.

Order-relevance, in contrast to the natural number counting approach, is constitutive for the understanding of numbers in the sense of the Stirling subversion.

A principle of *concretization*, in contrast to the homogenizing principle of abstraction, is essential for an understanding of numbers in a polycontextural sense.

1.3. Gunther's uncountable objects: Bad at math or bad math?

The situation today wouldn't be much different as it was for Gotthard Gunther when he asked, around 1908, his elementary school teacher two serious questions:

1. How is it possible that a simple addition of some single mountains (Berge) results into a mountain range (Gebirge)? That is, 5 Berge = 1 Gebirge, and how works that: 5 = 1!?

2. How is it possible to add different kinds of objects, like 1 church + 1 crocodile + 1 tooth pain + 1 thought together? And how would this relate to the example of the mountain rage (Gebirge)?

Would there be such a monster like a 'mountain-church-crocodile-tooth pain' range as a single, albeit complex object, like the addition of mountains is producing a mountain range?

The teacher's answer today will be more or less the same as a child got it a century ago.

As I just learned, a child today would have the chance to be tested by a Dyscalculia Screener about its math ability, and would have the honor to get a label for its deviation: *dyscalculia* or *acalculia*.

Hence, little Gotthard's courageous and intelligent rebellion against the abstractness of numeric addition wuld be conceived as part of a mental *"inability to conceptualize numbers as abstract concepts of comparative quantities".*

With dyscalcuia, qualified as a learning disability and classified as part of mental illness, all ways of discrimination of the child is opened up. Certainly, there are also some adults making a lovely business out of it.

http://www.ncld.org/types-learning-disabilities/dyscalculia/what-is-dyscalculia

Abstraction and enumeration (arithmetization) are just an adult answer that moves the question to another level of un-answered questions.

In his biographical text, *Selbstdarstellung im Spiegel Amerikas*, (1974) Gunther writes:

> *"Die Arithmetik mußte ganz anderes und Wunderbares leisten können, weshalb er an seinen Lehrer die Frage stellte: Wenn das Zusammensein von vielen Bergen ein Gebirge ergab, was ergäbe dann zahlenmäßig das Zusammensein, wenn man eine Kirche zu einem Krokodil addierte und dazu noch seine Mutter und obendrein ein Zahnweh. (Es ergab sich nämlich, daß gerade zu diesem Zeitpunkt seine Mutter an Zahnschmerzen litt.) Das erschien ihm als eine der Arithmetik würdige und hochinteressante Aufgabe.*
>
> *"Als man ihm mitteilte, daß man die vier angeführten Daten eben nur als verschiedene Sachen zusammenzählen könne, hielt er das zuerst für ein Mißverständnis und bestand darauf, daß er keine Sachen, sondern eben Kirchen, Krokodile usw. addieren wolle. Und was ändere sich am Addieren, wenn man das Krokodil durch einen Löwen ersetze? Daß sich dann nichts ändere, wollte er nicht glauben.*
>
> *"Später vergaß er das Problem. Er mußte fast 60 Jahre alt werden, bis es für ihn in der biologischen Computer-Theorie in neuer Gestalt wieder auftauchte."*
>
> http://www.vordenker.de/ggphilosophy/gg_selbstdarstellung.pdf

The development of Gunther's answers to his early questions went through several stages. From the kenogrammatic approach, to the polycontextural understanding of numbers and to a concept that is closely related to his theory of negative languages.

When Gunther started to answer his juvenile questions with his newly discovered kenogrammatics, his friend and colleague Heinz von Foerster was not just baffled but seriously concerned. He told Gunther quite directly: *"Gotthard, lass die Finger davon. Du hast da keine Ahnung."* This might be translated into the advise: "Gotthard stop it. You have no clue."

We have to thank Gotthard's stubbornness of 'astronomic dimension' that he didn't stop at all.

Today we might ask: Was Gunther *dyscalculic* or even *acalculic*? If yes, the defender of the dyscalculia syndrome have a world famous thinker on their side.

> *"There is no known way to prevent mathematics disorder."*
>
> *"Children who receive a diagnosis of mathematics disorder are eligible for an individual education plan (IEP) that details specific accommodations to learning."*
>
> Read more: http://www.minddisorders.com/Kau-Nu/Mathematics-disorder.html#b#ixzz2ivcPsWlp
>
> http://www.minddisorders.com/Kau-Nu/Mathematics-disorder.html#b

Gotthard Gunther's example makes it clear that there are possibilities to reject the wisdom of natural numbers that are in no way connected to any disabilities, handicaps or genetical depravations.

Gunther's question had been well articulated and the topic of numbers and shape recognition wasn't at doubt.

But their meaning was left in the dark. It wasn't explained to the pupil, and the teacher missed to touch the reasonable point of the pupil's question.

Hence, there is a 'acalculia' beyond 'discalculia'. It is called *trans-calculia*. This is the not yet well studied mental deviation that is surpassing the mathematical paradigm of natural numbers that defines the sane human mind set.

Some links to Gunther's relationship with numbers:

http://www.vordenker.de/ggphilosophy/gg_natural-numbers.pdf (1971)
http://www.vordenker.de/ggphilosophy/gg_number-and-logos_en-ger.pdf
http://www.vordenker.de/ggphilosophy/gg_identity-neg-language_biling.pdf (1979)

How does abstraction work and how are the natural numbers justified for such a counting process of different objects as young Gunther pointed out.

Again, we have the luck to ask Professor Philip Wadler from the university of Edinburgh. His answer is ultimate and should stop any such naive questions for ever.

In his lovely text, probably written for his children and some professors of computer science, Wadler makes it crystal clear:

> *"Whether a visitor comes from another place, another planet, or another plane of being we can be sure that he, she, or it will count just as we do: though their symbols vary, the numbers are* universal*.*

> *"The history of logic and computing suggests a programming language that is equally* natural*. The language, called lambda calculus, is in exact correspondence with a formulation of the laws of reason, called natural deduction. Lambda calculus and natural deduction were devised, independently of each other, around 1930, just before the development of the first stored program computer. Yet the correspondence between them was not recognized until decades later, and not published until 1980. Today, languages based on* lambda calculus *have a few thousand users. Tomorrow, reliable use of the Internet may depend on languages with logical foundations. "*
> Philip Wadler, As Natural as 0,1,2
> Evans and Sutherland Distinguished Lecture, University of Utah, 20 November 2002.
> http://homepages.inf.ed.ac.uk/wadler/papers/natural/natural3.pdf

Gunther was well aware that his kind of thinking, and his special way of understanding numbers, made him an Alien.

Not enough, in his late years he started to develop a system of arithmetics that not only answered his early two crucial questions but will also be enjoyed by Alien intelligence.

He sincerely told his baffled longtime friend Helmut Schelsky that he isn't anymore a human being, he just looks like one.

Also the discovery of the zigzag movement of numbers in a transclassical number system is amazing, it would be a sign of a serious lack of understanding Gunther's attempts towards a 'dialectical' number theory to celebrate this zigzagging against the '*Gänsemarsch*' of linearly ordered natural numbers as the sole achievement of Gunther's polycontextural constructions of the relation of 'number and logos'.

What could we learn today from this story?

Some primitive questions are not necessarily an expression of a lack of rationality but often more a sign or symptoms of another, still hidden, pattern of thinking and understanding the world.

Instead of destroying it, a teacher should be able to accept this 'deviant' way of thinking and be able to set it into a broader framework of different kinds of rationality.

Talking to the child and developing together new experiences could lead to surprising insights, relevant for the teacher and the curriculum too. It is a crime of the teachers and the government to deny the child such chances. Abuse has many faces. One still has to be unmasked.

1.4. Math for young dancers: Gaps and Jumps

1.4.1. Gaps and Jumps

Where in all those mathematical concepts of successor functions, induction steps, recursion cycles and deduction trees are the *gaps* and *jumps* that are natural to dancers?

It surely would be crazy if our abstract numerical counting process would have to stop somewhere at an obstacle, or falling into a counting gap or would have to jump out of such a paradoxical situation.

Why to trust in continuity?

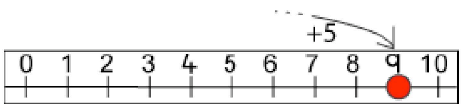
Also I was never a dancer I believe that life without gaps and jumps is grey. Personally, I was never convinced of this principle of homogeneous continuity necessary for induction, deduction and other step-wise developments of reasoning inside a single paradigm.

On the other hand, if we accept this principle of closure, life gets significantly boring and there is no special motivation to go into it.

Didactical jumps

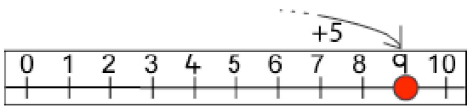
> *"First Leah made a jump of three along her number line and then a jump of four. Where did she land?*
> *"Next Leah made a secret jump along her number line. Then she made a jump of five and landed on 9.*
> *"How long was her second secret jump?*
>
> http://nrich.maths.org/5652

But an intriguing pre-mathematical question arises too: How does the child know on which number line the jump has to land? At least, there is more than one card on the table.



: number line One

↓



: number line Two

The classical supposition that there is one and only one arithmetical number line possible is not self-evident at all.

Why do we not have different number systems? Greens and reds and blacks?

As we know well, our teacher would explain us that all those differently colored number lines represent the same numbers because we can map each number from one color to the corresponding number of the other color. As they say, number systems are isomorphic. In color terms, they are all grey. And paradoxically, grey itself is not considered as a color.

Why should we accept that?

This principle of homogeneous continuity necessary for induction, deduction and other step-wise developments of reasoning inside a single paradigm has never got my enthusiasm.

On the other hand, if we accept this principle of closure, life gets significantly boring and there is no special motivation to go into it.

Also I was never a dancer I still believe that life without gaps and jumps is grey.

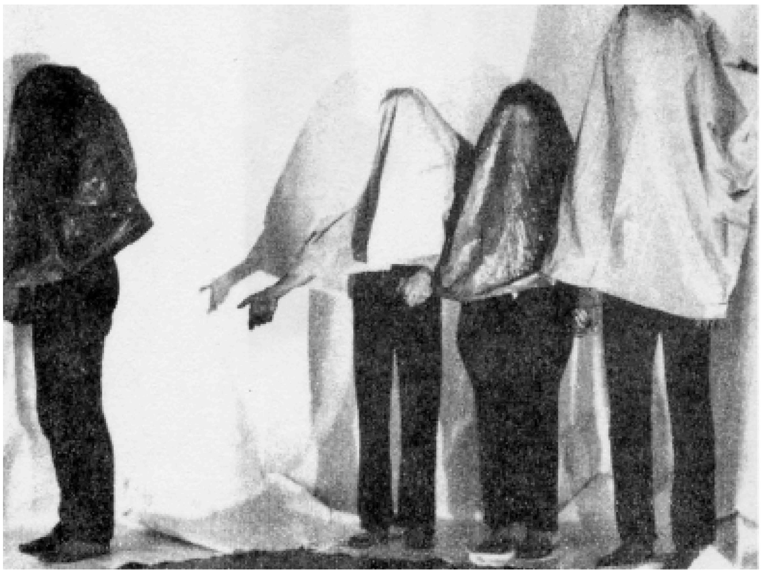
What do we learn from this not so innocent example of counting with number lines?

There are at least two different kinds of *jumps* possible: One *inside* a linear number system, and one *between* linear number systems.

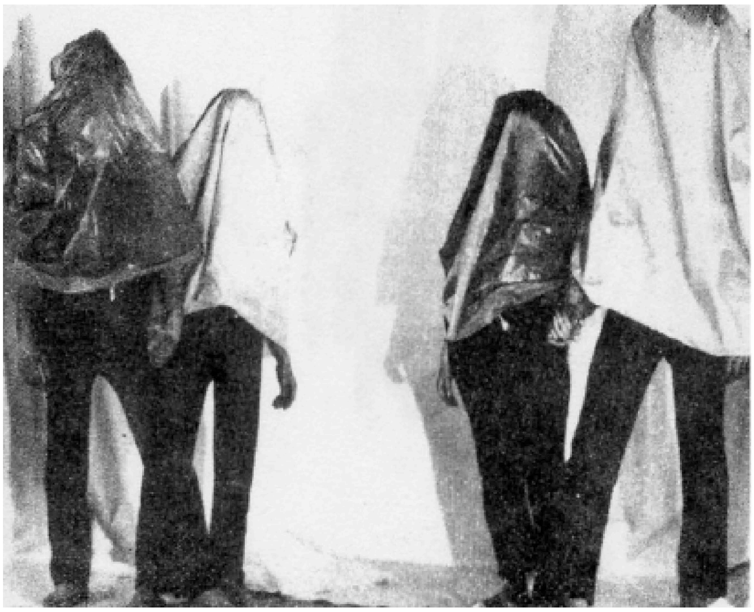
1.4.2. Choreography of Stirling numbers

Not all children prefer visual demonstrations of numbers. Some prefer movements and choreography. This happened at the famous Biological Computer Lab in Urbana, at the 15. May 1974, too. The abstractions necessary for an understanding of the Stirling numbers of length 4 had been presented by dancers from the class to their class as a little choreography of 7 different scenarios.

> *"Again: a partition of n objects is a division of these objects into separate classes. Each object must be in one and only one class and partitions with empty classes are not allowed. A question we might easily ask is how many ways can we partition n objects into k classes?"*



All 4 dancers are differently covered but the group is partitioned into 2 subgroups. One with 1 dancer and the other with 3 dancers, with [abbb]



The 4 dancers are partitioned into 2 different groups. Both two groups are in themselves different: [abab].

The 4 dancers are partitioned into 2 different groups. Both two groups are in themselves similar: [aabb].

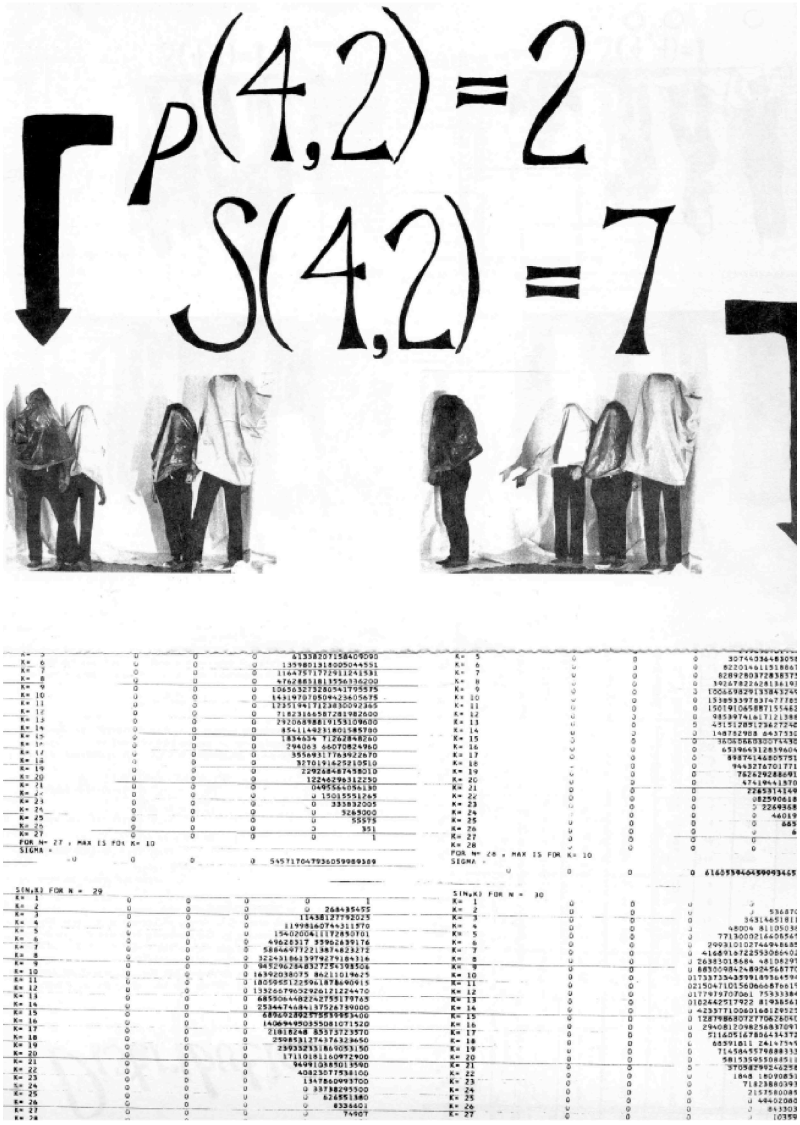
All 4 dancers are similar and are all together. There is just one Stirling number for such a constellation: [aaaa].

All 4 dancers are together. There are 2 groups of two similar dancers but they are alternately ordered: [abab].

All 4 dancers are separated and different by their performance. There is just one Stirling number for such a constellation: [abcd].

Hence, the sum of all Stirling numbers (of the second kind) for 4 dancers is just 15, i.e. Sum(1+7+6+1) partitions containing 1,2,3,4 sets. [aaaa],[aaab],[aaba],[abaa],[aabb],[aabc],[abaa],[abab],[abba], [abbb][abbc],[abca],[abcb],[abcc],[abcd].

For a partition in two sub-groups, there are just 7 realizations possiple.



http://memristors.memristics.com/CA-Overview/Short%20Overview%20of%20-Cellular%20Automata.pdf

2. Differences and Differentiations

2.1. Towards different differences

2.1.1. At school again

Imagine a school, where children are not forced to be students but are allowed to be curious and motivated for inquiring their own intelligence, their environment and their creativity and the creativity of other children.

Children in such a supportive environment are eager for knowledge about experiences of their own thinking abilities instead of focusing on perceiving properly some adult templates, like shapes, counting correctly small numbers and sorting things by elementary classification systems.

It seems that there are not many attempts to detect that are resisting the educational program of 'mathematizing' children's phantasy and creativity. And putting it onto a procrustean bed of identity.

What is 'mathematizing'?
The very simplest model for the production of natural numbers is based on the stroke calculus as a purely operative model of actions. Furthermore, natural numbers and their properties are seen as a standard model for even the most developed mathematical theories. The role of arithmetization is known as Gödelization after the famous German logician Kurt Gödel.

Therefore, we will have a glance at the famous *Stroke Calculus* presented in the 1960s by the mathematician Paul Lorenzen.

This is a very restricted model that is focused just on the repetition of an atomic sign.
Principles, like the positionality principle for natural numbers, are not yet reflected in this model. Nevertheless it demonstrates the essentials of the step-wise repetition of an atomic element on a line. And this is the very basic feature of mathematical thinking.

> *"The stroke calculus is ruling the way how to produce as many strokes as you want. To do that, it starts with the introduction rule R1 which allows to introduce one stroke as a start stroke. The second rule R2 rules how to produce from n strokes n+1 strokes. This is managed by an object variable n which doesn't belong to the production calculus but to its conditions. Thus, it is placed a Meta-Rules.*
>
> *"But the real point of the game is another rule which is mostly not mentioned at all: it is the indefinite iteration rule R3 which states that the production rule R2 can be applied as often as desired, i.e., potentially infinitely often. This is working together with the object variable which can deal with a set of potentially indefinitely many strokes."*

Stroke calculus
Rule1. $\Longrightarrow |$
Rule2. n $\Longrightarrow$ n |
Meta-Rule3. n $\in$ Var, repetition of Rule2.

Example
$\Longrightarrow |$: Rule1
$| \Longrightarrow ||$: Rule2
$|| \Longrightarrow |||$: Rule2, iteration

It is natural to establish a correspondence between the stroke-objects of the stroke calculus and *numerals*.
Hence, | corresponds to 1
|| corresponds to 2,
||| corresponds to 3, and so on.

It follows naturally to develop the rules of arithmetics, addition, subtraction, multiplication, and so on.

The laws of equality are also naturally introduced:

If | = | then n+| = n+|.

If n+| = n+| then |+ n = |+n

We shouldn't deny children the insight that the operative game is not as clean as it is proclaimed.

> *"An understanding of the "structure of the natural numbers" thus consists in an understanding of these rules. But what has actually been presented here? Rules R1 and R2 are fairly unambiguous, in fact, one could easily use them to write down a few numerals.*
>
> *"But rule R3 is in a different category. It does not determine a unique method of proceeding because that determination is contained in the words "apply R2 again and again".*
> *But these words make use of the very conception of natural number and indefinite repetition whose explantion is being attempted: in other words, this description is **circu**lar."*
>
> (Isle, p. 133), Epstein, Carnielli, Computability, p. 265/66
> http://www.tufts.edu/as/math/isles.html

From operations to dialogs
In a further turn, Lorenzen elaborated the *dialogical* aspect for basic math. This approach could help to develop a more 'natural' use of formal thinking for children than the purely monological operative understanding of math.

Therefore, as a new approach to introduce formal systems, Leibniz, Brown, Mersenne and Stirling, a dialogical setting and a dialogical game shall be chosen too.

This will be elaborated in a special chapter.

http://www.thinkartlab.com/pkl/lola/Games-short.pdf

2.1.2. Moshe Klein's approach based on George Spencer-Brown

An interesting project has been successfully realized in Israel: Moshe Klein's new approach to teach kindergarten children formal-mathematical thinking in a non-orthodox way based on dialogues between the teacher and the children.

> **Math Dialogue in Kindergarten**
> *"Our activity started in 1990 by developing an educational program in science for the kindergartens called "Reshit" (Genesis). The children are studying basic terms in sciences as: entropy, symmetry, movement and probability. The program is running in 1,200 kindergartens in 52 cities and settlements in Israel.*
>
> *"The main goal of this program, is to develop mathematics through a **dialogue** between the adult and the child." (M. Klein,*

Klein connects his approach with the insights of the *Calculus of Indication*, developed 1969 by George Spencer Brown.

In this case, children are focused on the *distinctions* represented by signs or brackets. It is observed that for two constellations (aa) and (bb) there are two different distinctions. That is (aa) is different from (bb). While for two constellations (ab) and (ba) there is just one distinction between "a" and "b". Therefore, the two constellations are considered as the same.

The basic rules for the Brownian distinction calculus

Rule 1. () () = ()
Rule 2. (()) = ∅
3. Substitution rules

Wording

Rule1: A distinction of 2 distinctions is a distinction.
Rule2: A distinction of a distinction is no distinction.

In colors

Rule1. ■ ■ = ■

Rule2. $\boxed{\blacksquare}$ = ∅

Other wording

Red with red saves red.
Red in red kills red.

The rules are also well understood as oriented actions.

Rule1. ■ ■ = ■ is an equational notation for to corresponding actions:
Rule1a. ■ ■ ⟹ ■ and
Rule1b. ■ ■ ⟸ ■

Rule2. $\boxed{\blacksquare}$ = ∅ this also holds for Rule2

Rule2a $\boxed{\blacksquare}$ ⟹ ∅

Rule2b. $\boxed{\blacksquare}$ ⟸ ∅.

Some examples

( ) ( ) () = ( ) : rule1 : ■ ■ ■ = ■
(( )) ( ) = ( ) : rule2, rule1 : $\boxed{\blacksquare}$ ■ = ∅

Proof of ■ ■ ■ = ■

[■ ■] ■ :brackets
[■] ■ : rule1
■ : rule1

■ [■ ■] :brackets
■ [■] : rule1
■ : rule1
Hence, the equation ■ ■ ■ = ■ holds.

Nobody said that 3 red apples are just one red apple.

It says: To draw a distinction and to repeat it, ■ ■, is to draw a distinction ■. This distinction ■, together with the third distinction ■ of the constellation ■ ■ ■,

repeats the previous situation, ■ ■. Hence, to repeat a distinction ■ ■, is to draw a distinction ■.
Therefore, to draw a distinction and to repeat it twice is equivalent to draw a distinction.
Hence, ■ ■ ■ = ■.

But we might compromise in the following wording:
To decide to eat the apple and to decide again to eat the apple and again to decide to eat the (same) apple again means nothing else, at least in a Brownian world, than to decide to eat the apple.

Thus, the Brownian universe is not about objects but about *decisions* and *distinctions*.

Georg Spencer-Brown started his Laws of Form 1969 with the command: *Draw a distinction! Mark it!*

In other words:
Red with red and red saves red .and. red and red with red saves red.
■ (■ ■) = (■ ■) ■ = ■.

Especially:

(( )) ( ) = ( ) (( )) : $\boxed{\blacksquare}\ \blacksquare = \blacksquare\ \boxed{\blacksquare}$.

Red in red kills red, $\boxed{\blacksquare}$ =∅ and red ■ saves red ■

equal

red ■ with red in red $\boxed{\blacksquare}$ kills red $\boxed{\blacksquare}$ =∅ and saves red ■.

Hence, $\boxed{\blacksquare}\ \blacksquare = \blacksquare\ \boxed{\blacksquare} = \blacksquare$.

Superpositions

(() ((()))) = $\boxed{\blacksquare\ \boxed{\boxed{\blacksquare}}}$

(() ()) = $\boxed{\blacksquare\ \blacksquare}$

(()) = $\boxed{\blacksquare}$

(()) = ∅

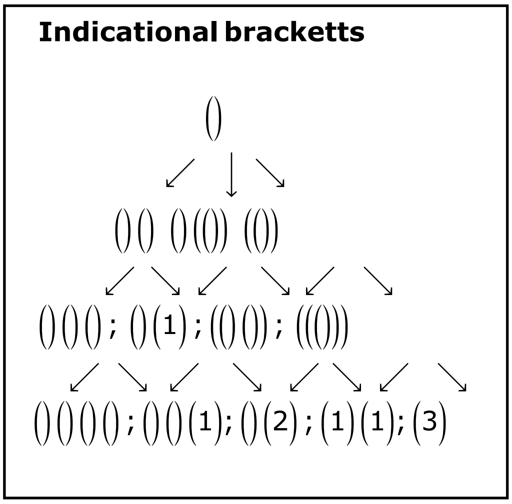
In words:

$\boxed{\blacksquare\ \boxed{\boxed{\blacksquare}}}$: Red with red in red in red kills red and red saves red: $\boxed{\blacksquare\ \blacksquare}$.

$\boxed{\blacksquare\ \blacksquare}$: Red with red in red saves red : $\boxed{\blacksquare}$.

$\boxed{\blacksquare}$: Red in red kills red ∅ .

For the teacher:



Moshe Klein published on Aug 12, 2013: "Forms of Numbers"
"During the last 20 years I have developed in kindergartens a dialogue approach to Mathematics. Thanks to that approach children are free from the hidden assumptions of adults concerning the nature of mathematics, e.g. that a line is composed of points. Recently I discovered by listening to young children a new concept which I call "Forms of Numbers":

"Consider two circles that you need to locate so they will not intersect with each other. It is easy to see that we have exactly two possibilities: circle near circle () () or circle inside circle (()). If we have three circles then we obtain already four possibilities. Circle near circle near circle ()()(); circle near circle inside circle () (()); circle with two circles inside which do not intersect (() ()); circle inside circle inside circle ((())).

"Children recognize that in the case of 4 circles there are 9 forms. This recognition shows that the children are already capable to distinguish between the difference of forms all of which are built with one symbol. This ability can be compared with the ability of listening, to which it is analogous.

"According to George Spencer Brown's "Laws of Form" (1969) binary and Boolean logic are only particular cases of using these circles, which create a new mathematical language by using only one operation he names distinction that requires only one symbol.

"This discovery supports Leibniz's vision developing a new language which will be more flexible than the standard logic where only two alternatives are possible, namely true or false. It might even support his first model of how controversies can be settled."

Workshop on Listening and Controversies: 23rd World Congress Philosophy. Athens, 4 August 2013
http://www.youtube.com/watch?v=NxQDRRKUEmY

Also this project is still preparing children for ‘adult-math’, it is far from taking an abusive parrot approach of the standard curriculum of education and training.

Doron Shadmi, Moshe Klein, *Various Degrees of the Numbers’ Distinction* http://www.omath.org.il/image/users/112431/ftp/my_files/VDND9.pdf

"We came to the conclusion that kindergarten children have a different way of grasping concepts and a different way of thinking than do adults. While the so-called "adult Mathematical thinking" is based mostly on Logic, children think in a way that is balanced somewhere in-between logic, intuition, emotion and imagination.

*"We believe that kindergarten is the natural environment for a growing mind to be trained to think **parallel** AND **serial** simultaneously, where Parallel thinking is more intuitive and Serial thinking is more analytical."*

http://scireprints.lu.lv/121/1/IJPAM-Volume-49.pdf

The idea to take the form of numbers into account deliberates numbers from representing a number of objects.

The number 5 e.g. may represent 5 identical objects. That's the classical approach.
But the counting number 5 may be seen as a form of partitions of the mathematical number 5:
"5" = {5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1}.

With this representation, 5 represents the set of collections from 5 identical to 5 different objects.

Therefore the number *5* in this context has two separate meanings: one as a *counting* number, and one as a *form* of a number. This distinction is not yet directly involved into the distinction of *cardinal* and *ordinal* numbers.

$\{■, ▲, ✱, ♠\} \Longrightarrow \boxed{5} \Longrightarrow_{\text{Brown}} \{5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1,$
$1+1+1+1+1\}.$

Philosophical speculations

"To sum up: numbers appear to represent both an attribute of matter and the unconscious foundation of our mental process."

"When we take into account the individual characteristics of natural numbers, we can actually demonstrate that they produce the same ordering effects in the physical and psychic realms; they therefore appear to constitute the most basic constants of nature expressing unitary psycho-physical reality." (Marie-Louise von Franz, 9. p.303)

Leibniz: Cardinality: 5 has 5 repesentations

```
- Pcontexture 5;
val it = [[1,1,1,1,1],[1,1,1,1,2],[1,1,1,2,3],[1,1,2,3,4],[1,2,3,4,5]] : int list list
```

Brown: Form: 5 has 7 representations

```
- Dcontexture 5;
val it =
```

[[1,1,1,1,1],[1,1,1,2,2],[1,1,1,1,2],[1,1,2,2,3],[1,1,1,2,3],[1,1,2,3,4],[1,2,3,4,5]] : int list list

Mersenne: Differentiation: 5 has 31 reperesentations

Mcontexture 5 = 31

[[1,1,1,1,1] , ...(31).., [1,2,3,4,4]]

Stirling: Pattern: 5 has 52 representations

```
- Tcontexture 5;
val it =
 [[1,1,1,1,1],[1,1,1,2,2],[1,1,2,1,2],[1,1,2,2,1],[1,2,1,1,2],[1,2,1,2,1],
  [1,2,2,1,1],[1,2,2,2,1],[1,2,2,1,2],[1,2,1,2,2],[1,1,2,2,2],[1,1,1,1,2],
  [1,1,1,2,1],[1,1,2,1,1],[1,2,1,1,1],[1,2,2,2,2],[1,1,2,2,3],[1,1,2,3,2],
  [1,1,2,3,3],[1,2,1,2,3],[1,2,1,3,2],[1,2,2,1,3],[1,2,2,3,1],[1,2,3,1,2],
  [1,2,3,2,1],[1,2,1,3,3],[1,2,3,1,3],[1,2,3,3,1],[1,2,2,3,3],[1,2,3,2,3],
  [1,2,3,3,2],[1,1,1,2,3],[1,1,2,1,3],[1,1,2,3,1],[1,2,1,1,3],[1,2,1,3,1],
  [1,2,3,1,1],[1,2,2,2,3],[1,2,2,3,2],[1,2,3,2,2],[1,2,3,3,3],[1,1,2,3,4],
  [1,2,1,3,4],[1,2,3,1,4],[1,2,3,4,1],[1,2,2,3,4],[1,2,3,2,4],[1,2,3,4,2],
  [1,2,3,3,4],[1,2,3,4,3],[1,2,3,4,4],[1,2,3,4,5]] : int list list
```

2.1.3. A Mersenne based approach

"You cannot teach a man anything; you can only help him to find it within himself." Galileo

Following Galileo's advise, a teacher might discover that a student of whatever age is developing a very different approach of thinking. Different from the classical identity approach but also different from the distinctional strategy of Klein's approach based on the Calculus of Indication.

It might turn out that with some help the student is developing a cognitive framework or paradigm that is unconsciously ruled mainly by the rules that constitutes the Mersenne calculus.

A child might focus on the *differentiations* of sign in a constellation and not on the distinctions (Brown) but also not on the differences between sign, like for the Stirling approach.

The result is, that for such a focus on the perception of differentiation, two constellations (aa) and (bb) are seen as lacking any differentiation, while two constellations, like (ab) and (ba) are showing a differentiation. Thus, (aa) and (bb) are seen as the same, while (ab) and (ba) are seen as different.

If children might be deviant to the classical identity-driven approach and choosing the *Brownian* distinctional rule, it shouldn't be astonishing if children are deciding for a 'complementary' and dual approach to the Brownian: the *Mersenne* rules.

And obviously, it is even less astonishing, if children are choosing freely between all 3 mind sets: Leibniz, Brown, Mersenne. Depending on the experienced task and its context or environment.

Only classically educated identity-maniacs would fear for the mental health of their children if they start to choose freely their own approach.

For a teacher to be able to support children on their own way of thinking, he/she/it has to be free enough to understand the different possible approaches or to learn to be able to discover them.

The 4 discussed approaches in this paper are not exclusive at all. At first there is the possibility given to find an interesting mix of the approaches. And certainly, someone will come up with a new and different approach too.

How is the Mersenne calculus defined? How does this *Calculus of Differentiations* differ from Brown's *Calculus of Indication*?

I shall paraphrase Klein's wording for the 'complementary' Mersenne approach: Consider two circles that you need to locate so they will not intersect with each other. It is
easy to see that we have exactly two possibilities of making a *differentiation*:
circle near circle () () or
circle inside circle (()).

The basic rules of the calculus of differentiations

Rule 1. () () = ∅
Rule 2. (()) = ()
3. Substitution rules

Wording

Rule1: A differentation between 2 differentiations is an absence of a differentiation.
Rule2: A differentiation of a differentiation is a differentiation.

A more suggestive wording might use the quotation concept:
Rule1: *A repetition of a quotation is the absence of a quotation.*
Rule2: *A quotation of a quotation is a quotation.*

In colors

Rule1. $\blacksquare\ \blacksquare = \varnothing$

Rule2. $\boxed{\blacksquare} = \blacksquare$

Other wording

Blue with blue kills blue.
Blue in blue saves blue.

The rules are also well understood as oriented actions.

Rule1. $\blacksquare\ \blacksquare = \varnothing$ is an equational notation for to corresponding actions:
Rule1a. $\blacksquare\ \blacksquare \Longrightarrow \varnothing$ and
Rule1b. $\blacksquare\ \blacksquare \Longleftarrow \varnothing$

Rule2. $\boxed{\blacksquare} = \blacksquare$ this also holds for Rule2

Rule2a $\boxed{\blacksquare} \Longrightarrow \blacksquare$

Rule2b. $\boxed{\blacksquare} \Longleftarrow \blacksquare$

But according to the basic rules of Mersennian differentiation we have three circles on level 2 equivalent to the Brownian calculus but we obtain not 4 but already 7 possibilities on the level 3 for the Mersenne calculus. Certainly, there are interesting consequences for a definition of plagiarism involved with those rules.

Also the Mersenne Calculus, I introduced some years ago, is considered as a complementary calculus to the Brownian calculus, it hasn't got the attention it deserves. Nevertheless, its rules are as transparent and intuitive as the Brownian rules.

Some examples

1. ()()() = (()(())(()) : the same are the same, thus there is no differentiation.

$\blacksquare\ \blacksquare\ \blacksquare = \boxed{\blacksquare}\boxed{\blacksquare}\boxed{\blacksquare}$

$\blacksquare\ \blacksquare\ \blacksquare = \blacksquare$

$\boxed{\blacksquare} = \blacksquare$

Thus, $\blacksquare\ \blacksquare\ \blacksquare = \boxed{\blacksquare}\boxed{\blacksquare}\boxed{\blacksquare}$.

2. ( ) ( ) () = () : rule1 : $\blacksquare\ \blacksquare\ \blacksquare = \blacksquare$

(( )) ( ) = ∅ : rule2, rule1 : $\boxed{\blacksquare}\ \blacksquare = \varnothing$

((())) = () : rule2 : $\boxed{\boxed{\blacksquare}} = \blacksquare$

Especially

3. (( )) = ( ) () () : $\boxed{\blacksquare} = \blacksquare\ \blacksquare\ \blacksquare$

Proof of $\boxed{\blacksquare} = \blacksquare\ \blacksquare\ \blacksquare$

[■ ■] ■ : brackets
[∅] ■ : rule1
■ : rule1.

$\boxed{\blacksquare}$

■ : rule2

Hence, ■ = ■, thus the constellation $\boxed{\blacksquare} = \blacksquare\ \blacksquare\ \blacksquare$ holds.

In words
Blue in blue saves blue as blue with blue and blue kills blue.

Comparison
Interestingly, there are some coincidences between both calculi. Both are deducing form the 3 brackets one resulting bracket: ( ) ( ) () = ().
But the way they are doing it is differently organized according to the 2 different rule sets.

It is a common failure to not to recognize this crucial difference.

Mersenne : ( ) ( ) () = () :

by rule1 :

■ ■ ■ = ■ :

(■ ■) ■ = (∅) ■ = ■

(■ (■ ■) = ■ (∅) = ■

Hence, ■ ■ ■ = ■ .

Brown: ( ) ( ) () = () :

by rule1 :

■ (■ ■) = ■ (■) = ■

(■ ■) ■ = (■) ■ = ■

Hence, ■ ■ ■ = ■.

In **contrast**:

$\boxed{\blacksquare}$ = ■ ■ ■

$\boxed{\blacksquare}$ = ■

■ ■ ■ = ■

Hence, $\boxed{\blacksquare}$ = ■ ■ ■.

$\boxed{\blacksquare}$ ≠ ■ ■ ■

$\boxed{\blacksquare}$ = ∅

■ ■ ■ = ■

Thus, ∅ ≠ ■.

Mixed calculi

Rule1. ■ ■ = ∅ : Brown

Rule2. $\boxed{\blacksquare}$ = ∅ : Mersenne

With an unspecified emptiness, like Mersenne and Brownian ∅, we could speculate a mix of calculi.

■ ■ = $\boxed{\blacksquare}$ = ∅.

But strictly, both 'emptiness' are different and the mix might not reasonable. Therefore, both too should be colored: Mersenne ∅ , Brown ∅, with ∅ ≠ ∅ and the equivalence "≠" belonging to a meta-language.

For the teacher:

Mersenne brackets

$$
\begin{array}{c}
()\\
\swarrow \quad \downarrow \quad \searrow\\
()() \qquad ()(()) \qquad (())()\\
\swarrow \ \downarrow \ \searrow \qquad \swarrow \ \searrow \qquad \swarrow \ \searrow\\
()()();\ (())();\ ()()(());\ ()(())();\ (()());\ (())()();\ (())()(())
\end{array}
$$

2.1.4. A Stirling based approach

Common grounds

What is the common presumption of the 3 different approaches to thinking and cognitive orientation in a world defined by those approaches?

The answer is simply given by the "*non-overlapping*" rule.
Consider two circles that you need to locate so they will *not intersect with each other.*
In other words, the presumption is the stability of the 'length' of the words. Numbers or words are not changing their length in the process of interaction or manipulation.

This presumption holds for the fourth, i.e. the morphogrammatic approach, too. But not necessarily. It is just a special, albeit common case. Morphograms are allowed to overlap and to merge with other morphograms without loosing their (own) rationality. But this is certainly another story.

Still following Kleins's advise to use brackets as a vehicle of notation, a further step in the game is possible.

From the Brownian rules, with e.g. (())() = ()(()) and the Mersenne rule, with (()) = (), we shall *blend* both approaches together with the new rules: (())()() ≠ ()(()) () and () = (()).

Blending concepts

> *"Blending two conceptual spaces yields a new space that combines parts of the given spaces, and may also have emergent structure."* (Goguen)

$$
\begin{tikzcd}
 & \text{Blend B : } \textbf{Stirling} & \\
\textbf{Mersenne} : \text{Input } I_1 \arrow[ur] & & \text{Input } I_2 : \textbf{Brown} \arrow[ul] \\
 & \text{Space G : } \textbf{Leibniz} \arrow[ul] \arrow[ur] \arrow[uu] &
\end{tikzcd}
$$

http://memristors.memristics.com/Dissemination%20as%20Blending/Dissemination%20as%20Blending.html

In other words, a child might be bored of distinctions and differentiations. It might feel that there is something beyond identification, distinction and differentiation.

It manifests as the thinking that two constellations, (a) and (b), are the same. But also the distinctional and the differential (ab) and (ba) are conceived as the same. And obviously, (aa) and (bb) are seen as just the same.
But something new happens with patterns with at least 3 places and 3 elements.

Also it will not be easy to become aware and clear with the new situation, it might just turn out that is this new constellation is grasped with the following of non-equivalences: (aaa) ≠ (aba) ≠ (abb) ≠ (abc).

Hence the Stirling blend might taste as a 'water of life' with 4 basic ingredients.

But with that, everything gets confused. All is lost. The chaos reins.

Teachers are pupils and pupils teachers.

Equality and order is refused.

But a new order appears.

It might be recognized that a constellation of 2 pupils and 1 teachers is different from a constellation of 1 pupil and 1 teacher and 1 pupil. Also there is a difference between those constellations and the constellation with 1 pupil and 2 teachers, and 1 pupil and 1 teacher and 1 head master.

But the constellation of 2 pupils and 1 teacher is perceived as the same as 2 teachers and 1 pupils. Etcetera.

Maybe this kind of carnival is not anymore accepted by the educational authorities for children to celebrate Hallowe'en 2013.

A Stirling blend

For a Stirling approach the fact that the concept of *patterns,* i.e. ordered strings of elements*,* is crucial, leads to the following rules.
Rule1. () = (())
Rule2. () () = (()) (())
Rule3. () (()) = (()) ()
Rule4. ()()(()) ≠ ()(())() ≠ ()(())(()) ≠ ()(())((())).

In colors

Rule1. ● = ■
Rule2. ● ● = ■ ■
Rule3. ● ■ = ■ ●
Rule4. ● ● ■ ≠ ● ■ ● ≠ ● ■ ■ ≠ ● ■ ○.

Another setting:

Rule1. ● ≡ $\boxed{\bullet}$

Rule2. ● ● = $\boxed{\bullet}$ $\boxed{\bullet}$

Rule3. $\bullet\ \boxed{\bullet} = \boxed{\bullet}\ \bullet$

Rule4. $\bullet\bullet\ \boxed{\bullet} \neq \bullet\ \boxed{\bullet}\ \bullet \neq \bullet\ \boxed{\bullet}\ \ \boxed{\bullet} \neq \bullet\ \boxed{\bullet}\ \boxed{\boxed{\bullet}}$.

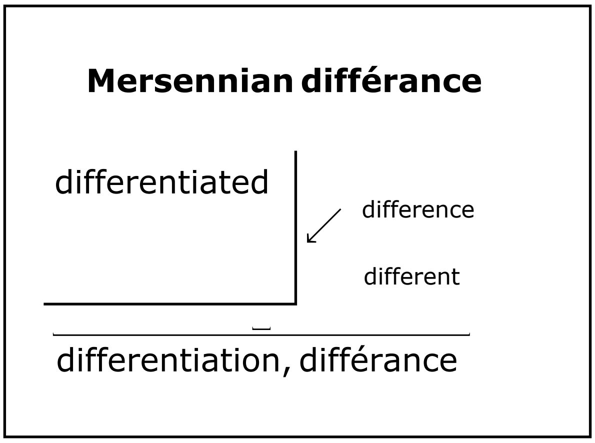
Comments
Substitution has to respect the pattern structure of the morhograms.
Also the commutativity of Rule3 holds, it is wrong to deduce:
●■● = r3(●●■) with (● r3 (●■)) = ●■● or to deduce with Rule2 by reduction
●●● = r2(●■■) with (● r2 (●■)) = ●●●.

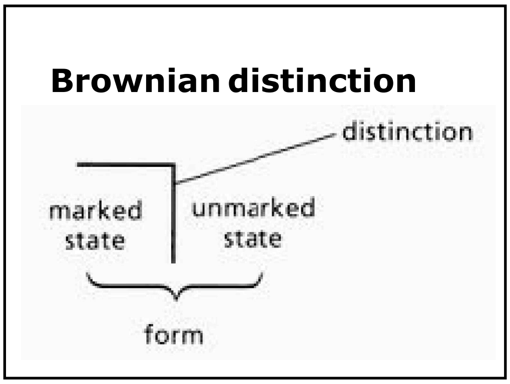
2.2. A metaphor for all 4 approaches

A metaphor to support teacher to get aware about the possible different kind of rationality might be given by the different decision strategies of a doorman at a Glaswegean night club.

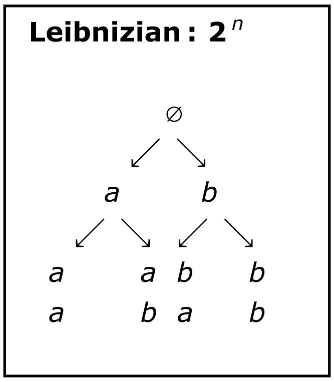
After a serious study of this little scheme of decision types and logics, and some personal experiences at the doors of different clubs, the teacher might be fit at the morning after to encounter the complex types of creativity of his/her kindergarten class.

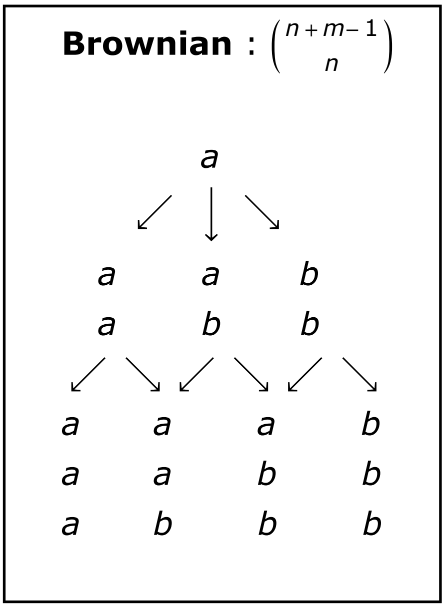
2.2.1. Diagrammatics of the 4 approaches

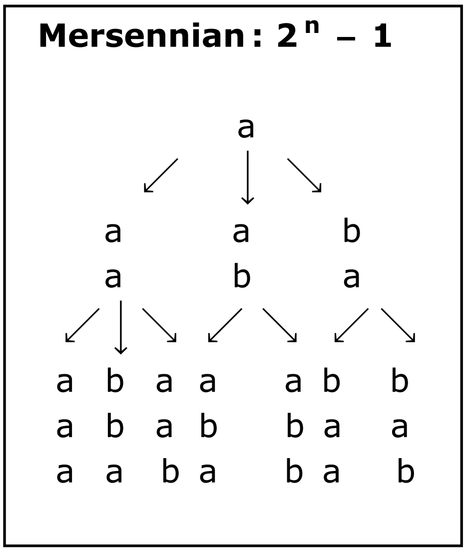




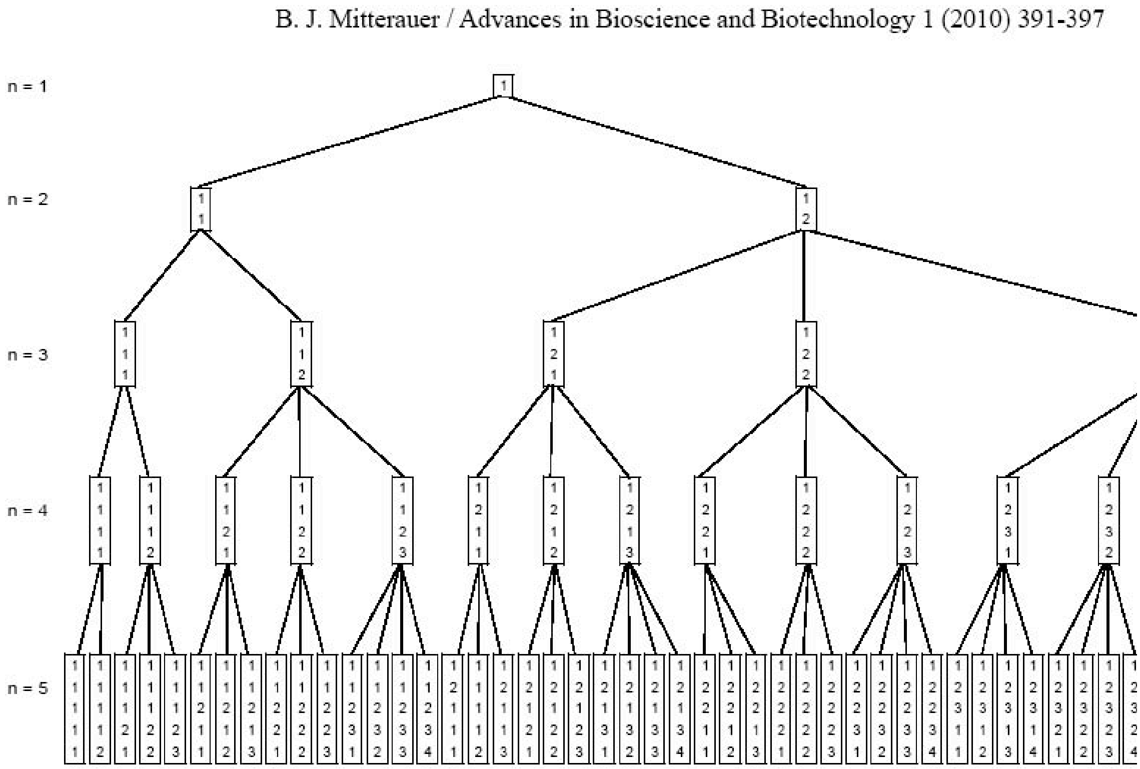
The four graphs for the two elements "a" and "b"







Bernhard *J*. Mitterauer has plottet some steps, up to 5, of the successor operation for the **Stirling** successors.

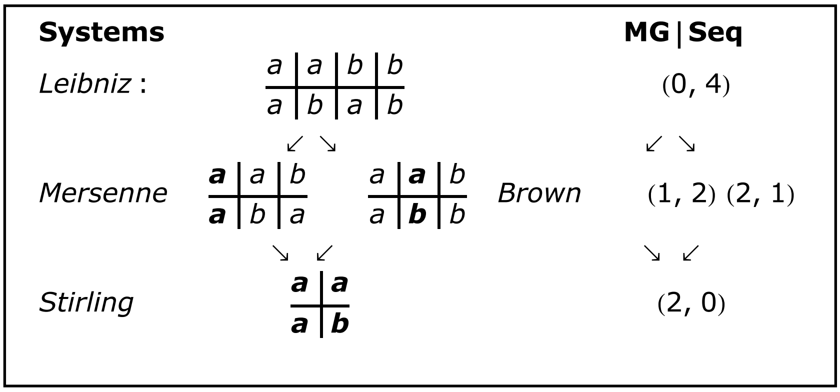


> *"The structure for tritograms with length 1 to 5 (5 levels) is represented by a tree. This is the generation rule: a tritogram x with length n+1 may be generated from a tritogram y with length n if x is equal to y on the first n places, e.g. 12133 may be generated from 1213 but not from 1212.*
>
> *"The numerals are representations of domains (properties, categories) that should be viewed as 'place-holders' reserved for domains, e.g. 12133 should be read as five places for five entities, such that the first and the third entity belong to domain one, the second entity to domain two, and the fourth and fith entity to domain three." (B. Mitterauer)*

It is difficult to find in Mitterauer's wordings a hint to the *retro-grade recurrent* structure of the successors. It seems to be obvious that there is no succession from the tritogram 1212 to the tritogram 12133. Such a move would have to go by emanation from 1212 to 1213 and then from there to 12133.

Four types of diagrams: Trees and Graphs

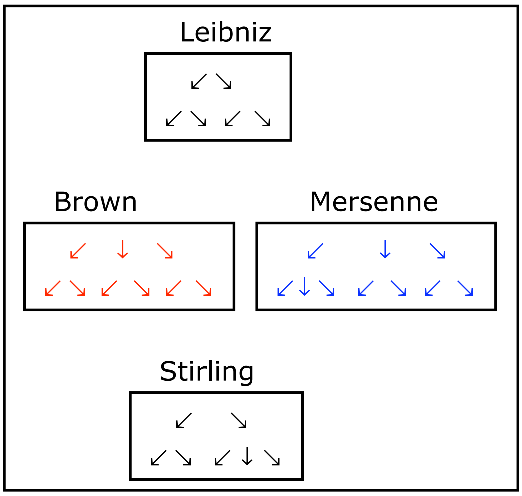
There might also be just some aesthetic reasons why a child is preferring a specific type of basic graphs of a general system of notation.



types\values	**aa**	**ab**	**ba**	**bb**	combinatorics
Leibniz	aa	ab	ba	bb	m^n
Mersenne	aa	ab	ba	–	$2^n - 1$
Brown	aa	ab	–	bb	$\binom{n+m-1}{n}$
Stirling	aa	ab	–	–	$\sum_{k=1}^{M} S(n, k)$

The Leibniz graph is surely a dyadic tree. This structure is very simple and easy to grasp. But it will quickly lose its attractiveness.

The Brownian graph is the only *commutative* graph of the 4 different types. A commutative graph is not easily to understand. There are states that are merging properties of different sorts.



With the application *MorphoGames*, it is good fun to play the 4 different types of graphs against each other, concerning some specific questions.

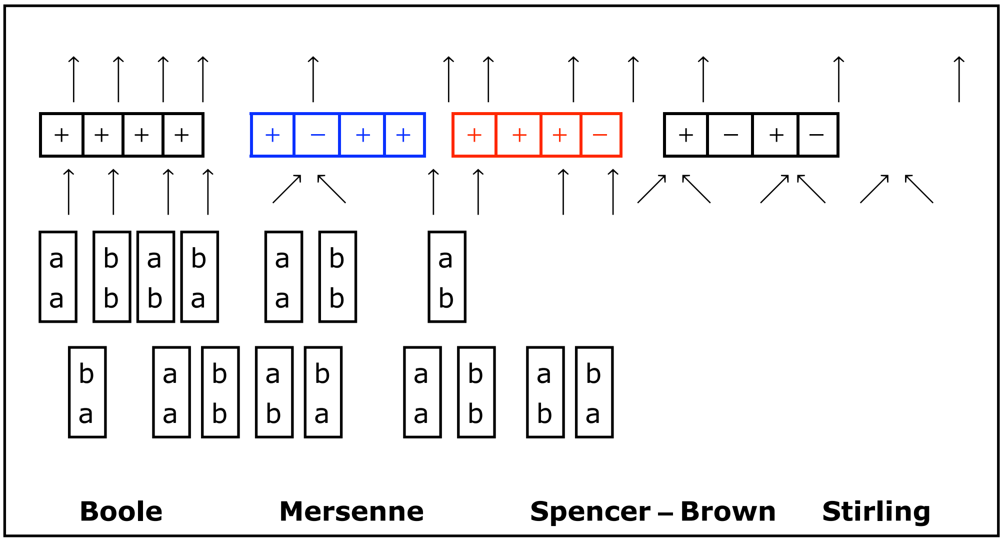
Gaps

Gaps appear in the interaction between different calculi. There is no direct access for a calculus to its own gap. Hence, a gap is a *blind spot* of a calculus. An *interactional* calculus of indication and differentiation is including the interactivity of calculi and gaps. Gaps are a third category to the "mark", "unmark" , ∅, and differentiation and absence of differentiation, ∅.

http://memristors.memristics.com/Complementary%20Calculi/Complementary%20Calculi.html
http://memristors.memristics.com/Interplay/Interplay%20of%20Elementary%20Graphematic%20Calculi.html

2.2.2. Decision strategies

Decision table for 4 strategies and 2 elements



Controller Example

A two-state controller (electronic, biological or human) with states {a, b} is configured to decide to accept (+) or to reject (-) a couple of signals {a, b} applying 4 different modes of acceptance/rejectance {Boolean, Mersennian, Brownian, Stirlingian}.

Boolean acceptance mode (++++):

the device accepts signals only iff they are distinct. Hence, the acceptance of the 4 elements is independent of their order, thus {a, b}, {a, a} and {b, b} are accepted.

Mersennian acceptance mode (+-++):

the device accepts signals only iff they are homogeneically the same (aa) = (bb) or permutatively distinct (a, b) ≠ (b, a). Hence, the acceptance of the 4 elements is depending on their order, thus (a, b) and (b, a) are accepted as different. And (a, a) is proceeded as equal to (b, b).

Brownian acceptance mode (+++-):

the device accepts signals only iff they are homogeneically distinct And permutatively the same (a, b) = (b, a) = {a, b). Hence, the acceptance of the 4 elements is independent of their order, thus {a, b}. And (a, a) ≠ (b, b) are recognized as different.

Stirlingian acceptance mode (+-+-):

the device accepts signals only iff they are structurally distinct. Hence, the acceptance of the 4 elements is independent of their order and their elements.

Hence (a, a) = (b, b) and (a, b) = (b, a). But (a, a) ≠ (a, b).

Glaswegian Bouncer Example

If this abstract example of a *controller* is not sufficient enough to satisfy the desire for examples, then it might be more entertaining to use the conflict-strategy interpretation of an ordinary Glaswegian Doorman at the doors of his club.

***Case one*:** Monday night, everyone is accepted (to pay entry and the drinks). Nevertheless, the clubbers are well recognized and distinguished as male and female. Hence, no one else beyond this distinction is accepted.

***Case two*:** Tuesday open night: Male and female only couples are accepted and treated as the same, i.e. with the same conditions in respect of dress code and ticket price. Mixed couples are accepted as different and treated differently depending on the dominance of one of the partners. Female dominated couples are preferred and are paying less. Hence, a lesbian girl with a guy (a, b) is preferred to a gay guy with a girl (b, a). Thus, there is no chance to get any promotion for same sex couples.

***Case three*:** Wednesday special night: Male only couples and female only couples are accepted but differently taxed. Mixed couples are accepted and treated equally. Hence, there is no chance for them to differentiate mixed couples and to get some reductions on drinks or entry.

Case four: Friday/Saturday night: The club has to be filled! All the differentiations are obsolete and there are no special reductions, promotions or bargains. Everything is cheap anyway. You just have to be from the suburbs to be accepted. Nevertheless there is still some differentation between same sex couples in general and mixed sex couples in general. Mixed sex couples have a chance for some support and a reduction on the toilet fee. While same sex couples might be preferred by the wardrobe girl.

***Case fife*:** Sunday night: Special effects, for insider only, not depending on any controller. Hence, there is even a chance to enter the club without being bound to a partner of whatever sex.

A two-bouncers decisions

But things are more complicated at the doors! Now, we get 2 bouncers at one door and their job is to fill the club with the same amount of people. The preferences for the decisions are free.

Hence, also the Mersennian and the Brownian bouncer are delivering the same results, their criteria of decision are strictly different and unknown to each other and the clubbers.

http://memristors.memristics.com/Graphematics%20of%20Conflicts/Graphematics%20of%20Conflicts.html

2.3. Teacher’s lesson: Addition, multiplication, palindromes

2.3.1. Classical approach: Sign sequences

Now back to school, a teacher might contemplate about his set of tools again. The existing building blocks, but also the established rules for the building blocks have lost their triviality.

A red and a green plastic stone is still a red and a green plastic stone. That's according to the classical approach. But does it matter anymore?

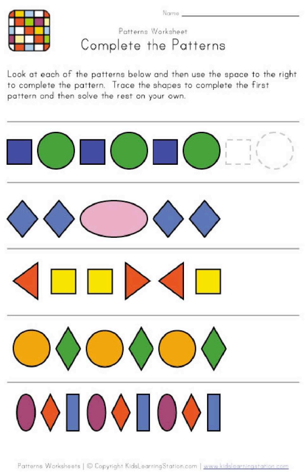
Classical basics at:
http://uk.ixl.com/math/reception

The classical approach is based on the *identity* of the elements and on the relevance of their position in a (linear) order.

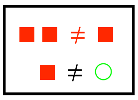
Numerical Mechanisms and Children's Concept of Numbers
http://web.media.mit.edu/~stefanm/society/som_final.html

Pattern recognition

> *"Help kids develop their early problem solving skills with this set of printable pattern worksheets."*
>
> http://www.kidslearningstation.com/preschool/pattern-worksheets.asp



Classical rules

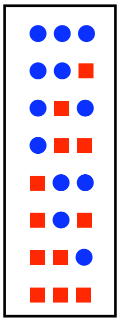


Wording
Two elements are not equal one element.
Different elements are different and not equal.

Given 2 elements and 3 places, how many different constellations of the two different elements on the 3 places are possible?

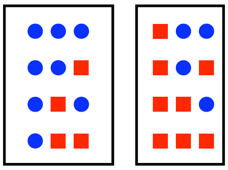
The Leibnizian order for 2 elements and 3 places has 8 constellations.

Leibniz(3,2) = 8:



Symmetry

There is also a nice symmetry between the first and the second half of the Leibniz patterns.



Classical topics

Standard forms

The main rule for the Leibnizian approach says that the sequences or pattern we see are the sequences and patterns we are dealing with. Certainly, this is a consequence of the rule of identity: (■ ≠ ○).

This might be called a Wysiwyg approach.

This convenient situation is disturbed in one or the other way with all 3 following approaches.

To be able to deal with the sequences or patterns from the Brownian, Mersennian and Stirlingian approach we have to build and accept so called *standard forms*. The standard forms are representing classes of possible realizations of the sequences and patterns.

Succession, addition, multiplication, reversion and palindromes

Succession, addition and multiplication of classical examples are well known.

Successor

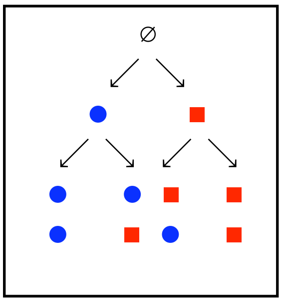
The number of successors depends on the number of elements in the alphabet. With an alphabet of one element we get one successor, with 2 element, we get two successors, and so on.

Alphabet ∑ = {●}
succ(●) = ● ●

Alphabet ∑ = {●, ■}
succ(●) = {● ●, ● ■}
succ(■) = {■ ■, ■ ●}

This successors are defining a *binary* tree. With 3 elements the successors are defining a ternary tree.

Binary tree for Leibniz



Reversion

As easy as additions are reversions of patterns with 4 elements.
(○ ■ ● ▲ ▲) = pattern
(▲ ▲ ● ■ ○) = reversion of the pattern.

Classical palindrome

More fun happens if we ask if a reverted pattern or sequence is still equal the original pattern.
If both are equal the pattern is *symmetric* and is called a *palindrome*.

A usual palindrome example is mentioned as “Anna”. It read forwards and backwards the same.

Another example: palin(8, 4) = (○ ■ ● ▲ ▲ ● ■ ○).

This pattern has the length 8 and consists of 4 different elements. It reads forwards and backwards the same. Thus, it is a palindrome.

The rules for the building of classical palindromes are easy to understand. If we add to a given element an additional element on the right and on the left side, we get a palindrome:

For an alphabet $\sum$ = {●, ■} we get:
● $\Longrightarrow$ ● ● ●, ■ ● ■
■ $\Longrightarrow$ ■ ■ ■, ● ■ ●.

The rules are given with this little grammar.

Alphabet :
$\sum$ = {●, ■}

Rules :
1. S $\longrightarrow$ ● S ●
2. S $\longrightarrow$ ■ S ■
3. S $\longrightarrow$ ε | ● | ■

Examples:

Odd palindrome

With rule 3 we introduce a start token, say ●. Now S is ●, and ● is palindrome.
Apply rule1 to S: S ⟶ ● S ●. Now S is ● ● ●, is palindrome,
Apply rule2 to S: S ⟶ ■(● S ●) ■. Now S is ■ ● ● ● ■, is palindrome.
Apply rule1 to S: S ⟶ ● S ●. Now S is ● ■ ● ● ● ■ ●, is palindrome.
And so on.

The order of the application of the rules rule1 and rule2 is free. The result is always symmetric, and therefore a palindrome. There are no surprises included in this parcel.

Even palindrome

A more interesting example is given with (○ ■ ● ▲ ▲ ● ■ ○).
The alphabet is: $\sum = \{$●, ■, ○, ▲$\}$ and a new
rule4: S ⟶ ▲ S ▲,
rule5: S ⟶ ○ S ○.
With rule 3 we introduce a start with the empty token ε. Now S is ε, ε is a nil-palindrome.
rule4: S ⟶ ▲ S ▲ ,
rule1: ▲ S ▲ ⟶ ● (▲ S ▲) ●,
rule2: ● ▲ S ▲ ● ⟶ ■ (● ▲ S ▲ ●) ■ ,
rule5: ■ (● ▲ S ▲ ●) ■ ⟶ ○ (■ (● ▲ S ▲ ●) ■) ○.
With rule3 we replace S by ε: thus we got the palindrome: ○ ■ ● ▲ ▲ ● ■ ○.

The same holds here. Free application of the rules, and no surprise in the box.

Test with MorphoGames

for palin(8, 4) = (○ ■ ● ▲ ▲ ● ■ ○).

- palindrome[1,2,3,4,4,3,2,1];
val it = true : bool

What did we learn?

Firstly, we can easily produce our own palindromes.
Secondly, to all possible classical (finite) patterns we can decide if they are palindromic or not.
Unfortunately things are in fact more complicated, because to apply the rules, we have to find the middle element of the sequence.

2.3.2. Brownian approach: Partitions

Partitions

> *"A partition of a number n is a way to present it as a sum of non negatives integer numbers when the order has no significant. An example to a partition of 3 is 3=1+2.*
>
> *"The partition function is the number of different partitions of a specific number n and it is written as p(n). The partition function is first mentioned in one of Leibniz's letters to J.Bernoulli (1674)." (Moshe Klein, Recursion over Partitions)*

The Brownian approach is based on the identity of its elements and the *irrelevance* of their position in the linear order. The position of the elements is commutative. Hence for any two elements "●" and "■" the concatenation results "● ■" and "■ ●" are equivalent in the Brownian universe.

More concrete examples for a *dialogical* math education for the kindergarten on the level of Brownian patterns is published with Klein's videos.

Again, the commutativity (( )) () = ( ) (()) is represented by different objects:

○■ = ■○
○○ ≠ ■■

Wording
In a Brownian universe, the order of 2 different elements is irrelevant. In contrast to the Mersennian universe, where they are different.
A group of two elements is equal to another group of two elements.
The same two elements are different to the same 2 different elements.

Alternative wording
Green and red together kills red and green.
Two greens together are safe with two reds.

Partitions for 4 elements and 4 positions
1+1+1+1: ()()()(): ● ● ● ● : aaaa
1+1+2: ()() (1): ○ ■ ● ● : aabc
(2)(2): (1)(1): ○ ○ ● ● : aabb
1+3: ()(2): ○ ● ● ● : abbb
4: (3): ○ ■ ● ▲ : abcd

The Brownian order for 2 elements and 3 positions has 4 patterns.
Brown(3,2) = 4

●●●
●●■
●■■
■■■

Brownian patterns are order-free, i.e. their elements are commutative, and are allowed to change position. Hence, Brownian palindromes are free under permutation.

Symmetry

There is a nice symmetry between the first and the second half of the patterns.

●●●
●●■

●■■
■■■

Successor

Recursion for Brown successor Succ

$\mathrm{Succ}(0) = 0$: R1
$\mathrm{Succ}(a) = \{aa, ab, bb\}$: R2.1, R2.2, R2.3
$\mathrm{Succ}(x) = \{xa, xb\}$: R3.1, R3.2
$\mathrm{Succ}(x) = \{\hat{x}a, \hat{x}b\}$: R4.1, R4.2

$\hat{x} = (x_i, x_j),\ i \neq j$
bnf(x) : Brownian normal form of x.

Examples
Successor
succ(●) = {(● ●), (● ■), (■ ■) }

Addition Sum

Sum(●, ∅) = ●
sum(● ●, ●) = {● ● ●, ● ● ■}.
sum(● ■, ●) = {● ■ ■}
sum(■ ■, ●) = {■ ■ ■}.

In a more formal setting

Sum(a, Succ a) = Succ(Sum(a, a))
= Succ(aa, ab, bb) = {aaa, aab; abb; bbb} : R2.x
with {aba, bba} ∉ bnf
Sum(a, Succ aa) = Succ(Sum(a, aa))
= Succ(aaa, aab, bba, bbb)
= {aaaa, aaab, bbba; aabb; bbbb}.
with {aaba, *aaba*, bbaa, *bbaa*, *bbab*, bbab} ∉ bnf
Sum(a, Succ ab) = Succ(Sum(a, aa))
= Succ(aa, ab, bb) = {aaa, aab; abb; bbb}.

Sum(a, Succ bb) = Succ(Sum(a, aa))
= Succ(aa, ab, bb) = {aaa, aab; abb; bbb}.

Multiplication Prod
Prod(a, 0) = 0
Prod(a, Succ 0) = Sum(a, Prod(a, 0)) = Sum(a, 0)) = a
= Prod(a, a) = a
Prod(a, Succ a) = Sum(a, Prod(a; aa, ab, bb)) = Sum(a, (aa, ab, bb))
= {aaa, aab; abb; bbb}.

Reversion for Brownian patterns

rev(● ■) = (■ ●) and (● ■) = (■ ●).

Partition based palindromes

Is the Brownian pattern [● ● ■] a palindrome?
Because of the standard normal form convention of Brownian patterns we know that
[● ● ■] $=_{\text{Brown}}$ [● ■ ●]. But, the pattern [● ■ ●] is palindromic.

That is, the standard form pattern [● ● ■] represents the set of equivalent patterns
{[● ■ ●], [■ ● ●]}.

Permutation of the classical palindrome example [1,2,3,4,4,3,2,1].

```
- ispalindrome(dnf[1,2,3,4,3,4,1,2]);
val it = true : bool
- ispalindrome[1,2,3,4,3,4,1,2];
val it = true : bool
- ispalindrome[1,2,3,1,2,3,4,4];
val it = false : bool
- dnf[1,2,3,1,2,3,4,4];
val it = [1,1,2,2,3,3,4,4] : int list
- ispalindrome[1,1,2,2,3,3,4,4] ;
val it = true : bool
```

2.3.3. Mersennian approach: Differentiations

The groups of *differentiations*, called *situations*, are defined by the Mersenne distribution of elementary differentiations with the combinatorial formula: 2^n-1.
Such groups are embedded into differential contexts.

○ ■ ≠ ■ ○
○ ○ = ■ ■

Wording

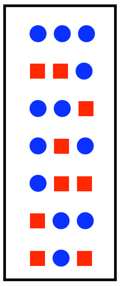
In a Mersenne universe, the order of 2 different elements is relevant. In contrast to the Brownian universe, they are different.
A group of two elements is equal to another group of two elements.

Alternative wording

Red and green together are safe.
Two greens together are killed by two reds.

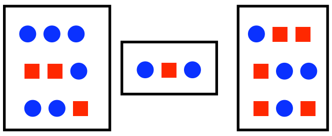
Constellations

The Mersenne order for 2 elements and 3 positions includes 7 patterns.
Mersenne(3,2) = 7



Symmetry

The nice symmetry of the whole set as we have seen before is broken.



A first interesting result of comparing the results of Leibnizian, Brownian and Mersennian approaches shows that Leibniz and Brown are symmetric in their basic constellations.
Mersenne, and as we will see, Stirling, are not anymore symmetric.

Recursion for Mersenne successor Succ

Succ(0) = 0 : R1
Succ(x) = {x^a, x^b, $\overline{x}$^a} : R2.1, R2.2, R2.3
Succ(x) = {xa, xb} : R3.1, R3.3

$\overline{x} = (x_i \ldots x_j),\ i = j$
mnf(x): Mersenne normal form of x.

Example

succ(●) = {(● ●), (● ■), (■ ●)}.

Addition Sum

sum(●, ∅) = ●
sum(● ●, ●) = {● ● ●, ● ● ■, ■ ■ ●}
sum(● ■, ●) = {● ■ ●, ● ■ ■}
sum(■ ●, ●) = {■ ● ●, ■ ● ■}.

In a more formal setting

```
Sum(a, 0) = a
Sum(a, Succ 0)  = Succ(Sum(0, a))
                = Succ(a) = {aa, ab, ba}.          : R2.x
Sum(a, Succ a)  = Succ(Sum(a, a))
                = Succ(aa, ab, ba) = {aaa, aab, bba; aba, abb; baa,
bab}.
Sum(a, Succ aa) = Succ(Sum(a, aa))
                = Succ(aaa, aab, bba),
```

```
= Succ(aaa) = {aaaa, aaab, bbba},   : R2.x
= Succ(aab) = {aaba, aabb},         : R2.1, R2.2
= Succ(bba) = {bbaa, bbab}.         : R2.1. R2.2

 Succ(aba, abb, bab),
 = Succ(aba) = {abaa, abab},         : R2.1. R2.2
= Succ(abb) = {abba, abbb}.         : R2.1. R2.2
= Succ(baa) = {baaa, baab},         : R2.1. R2.2
= Succ(bab) = {baba, babb}.         : R2.1. R2.2

Succ(abaa) = {abaaa, ababb},         : R2.1. R2.2
Succ(abab) = {ababa, ababb},         : R2.1. R2.2
```

Multiplication Prod

```
Prod(a, 0) = 0
Prod(a, Succ 0) = Sum(a, Prod(a, 0)) = Sum(a, 0)) = a
                = Prod(a, a) = a
Prod(a, Succ a) = Sum(a, Prod(a; aa, ab, ba)) = Sum(a, (aa, ab,
ba))
                = {aaa, aab, bba; aba, abb; baa, bab}
```

Comparison of Brownian and Mersennian calculi

Brown: `Sum(a, Prod(a; aa, ab, bb)) =`
`Sum(a, (aa, ab, bb)) =`
`{aaa, aab; abb; bbb}.`

Mersenne: `Sum(a, Prod(a; aa, ab, ba)) =`
`Sum(a, (aa, ab, ba)) =`
`{aaa, aab, bba; aba, abb; baa, bab}.`

Reversion for Mersenne

rev(ab) = (ba) and (ab) ≠ (ba)
rev(● ■) = (■ ●) and (● ■) ≠ (■ ●).

2.3.4. Stirlingian approach: Morphograms

Rules

a = b, ab = ba, aab ≠ aba ≠ abb ≠ abc

Stirling order with 3 elements and 2 positions for distribution:

○○	= ■■	= ●●	= ε
○■	= ■○	= ●○	= V
○●	= ●■	= ■●	= V

Because the order of the objects plays a role, morphograms have to have a length of at least 3 to make their behaviour transparent. Morphograms are morphic patterns, i.e. patterns where the identity of the objects doesn't matter. Such objects are called kenograms. Therefore they have to be written in standard normal form, that is by a freely chosen alphabet as a convention. What matters, in contrast to the Brownian costellations, is the position of their kenograms.

In the examples, blue (●) is chosen as the standard normal form, hence, patterns with other colors are equivalent to the blue form.

● ● ● ≡ ■ ■ ■ ≡ ○ ○○

Stirling order for 3 elements and 3 positions for distribution.

1^3: ● ● ● : aaa
$1^2 2^1$: ● ● ■ : aab
$1^1 2^1 1^1$: ● ■ ● : aba
$1^1 2^2$: ● ■ ■ : abb
$1^1 2^1 3^1$: ● ■ ○ : abc

Stirling (3,3) = 5

●●●
●●■
●■●
●■■
●■○

Hence, there are 5 different morphograms for 3 elements and 3 positions. The choice of the color of the elements, here as blue, red and green is arbitrary.

Wordings of constellations

For a Stirlingian game with 3 elements, some typical situations occur.

1. ● ● ● ≡ ■ ■ ■ ≡ ○ ○ ○,
   ● ● ■ ≡ ■ ■ ● ≡ ○ ○ ●
   etcetera
2. ● ● ■ ≡ rev(● ■ ■) : reversion
3. ● ■ ● ≡ rev(● ■ ●) : self-symmetry
   ● ● ● ≡ rev(● ● ●)|
   ● ■ ○ ≡ rev(● ■ ○)

A pattern of 3 blue elements kills the patterns of 3 red and 3 green elements. Three different elements are safe under permutation.

Wordings of rules for Stirling(3,3)

● ≡ ■: Blue kills red.
● ■ ≡ ■ ●: Blue together with red kills red together with blue.
● ● ■: Two blue together with one red, and
● ■ ●: one blue together with one red and one blue, and
● ■ ■: one blue with two reds, are safe in the Stirlingian world.
● ■ ○: As well as blue and red and green together.

This constitutes a kind of safety in groups.

Symmetry

Here, again, the symmetry of the set of the basic patterns is broken. But there are some nice internal symmetries left.

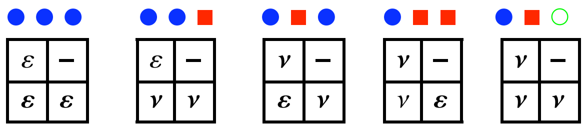
rev(● ● ■) = (● ■ ■), that is rev(● ● ■) = (■ ● ●) but (■ ● ●) = (● ■ ■).
Self-symmetric patterns: (● ● ●), (● ■ ●), (● ■ ○).

Difference notation with ν=non-equal and ε=equal

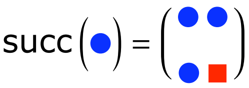
The fact that the presentation of the morphograms by specific elements is arbitrary has to be considered as crucial. Therefore, not the elements are determining the morphic patterns but the differences between the elements.

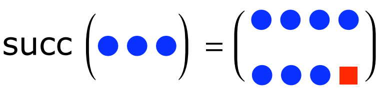
This is well depicted for the example [● ■ ●].

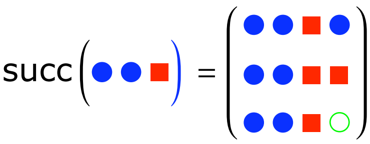
A useful notation is given with the matrix of the patterns.



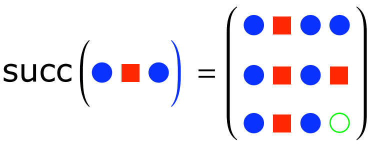
Successor

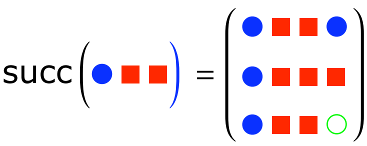


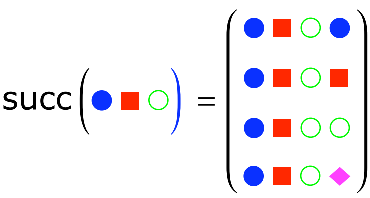




Null







Left successor

Example

$$\mathrm{l-succ}\left(\bullet\blacksquare\blacksquare\right)=\begin{pmatrix}\bullet\bullet\blacksquare\blacksquare\\ \blacksquare\bullet\blacksquare\blacksquare\\ \circ\bullet\blacksquare\blacksquare\end{pmatrix}=\begin{pmatrix}\bullet\bullet\blacksquare\blacksquare\\ \bullet\blacksquare\bullet\bullet\\ \bullet\blacksquare\circ\circ\end{pmatrix}$$

$$\mathrm{succ}\left(\bullet\blacksquare\blacksquare\right)\neq\mathrm{l-succ}\left(\bullet\blacksquare\blacksquare\right)$$

Addition

$$\mathrm{add}\left(\bullet\bullet,\bullet\blacksquare\right)=\begin{pmatrix}\bullet\bullet\bullet\blacksquare\\ \bullet\bullet\blacksquare\bullet\\ \bullet\bullet\blacksquare\circ\end{pmatrix}$$

Multiplication

$$\mathrm{kmul}\left(\left[\bullet\blacksquare\right]\left[\bullet\blacksquare\right]\right)=\begin{pmatrix}\bullet\blacksquare\blacksquare\bullet\\ \bullet\blacksquare\circ\bullet\\ \bullet\blacksquare\blacksquare\circ\\ \bullet\blacksquare\circ\blacklozenge\end{pmatrix}$$

Test with MorphoGames

-kmul[1,2][1,2];

val it=[[1,2,2,1],[1,2,3,1],[1,2,2,3],[1,2,3,4]]

Comparisons

Differentiations (Mersenne) are not differences (Stirling).

The constellations $\bullet\bullet\blacksquare$ and $\blacksquare\blacksquare\bullet$ are Stirling equivalent but Mersenne differentiated.

$\bullet\bullet\blacksquare$

ε	$-$
ν	ν

and

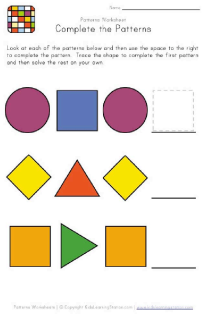
$\blacksquare\blacksquare\bullet$

ε	$-$
ν	ν

2.3.5. Revisiting the school examples for prolongations

All 3 examples of the sheet for prolongations are of the same morphogrammatic form.

The intended differences for the succession are based on the different shape and colors only. And obviously not on the differences between the elements.



The Kindergarten support team suggests that the only concrete and correct succession follows the alternating pattern of the examples.

For a Leibnizian approach, this is the only way to succeed. That’s well known by teachers, mothers and pattern recognition programs.

Hence, the prolongation follows the alternating structure of the given examples. For the first example, there are just two prolongations possible, the round red and the square blue. Following the alternating pattern of the example there is only one *correct* prolongation, the square blue. An involvement of the other patterns, yellow, red triangle, orange and green into the first solution is not intended.

But in a Stirling world, the differences are leading and not the patterns, and therefore the successors are depending not only on the elements but on the possible prolongations defined by the differences of the given pattern.

> *"In keno-writing, the number of kenoms to choose from as "the next letter" would be a dynamically changing variable: it equals the number of different kenoms used so far, plus one - because the next letter could always be a new one.”* (Rudolf Matzka, 1993)
>
> http://www.rudolf-matzka.de/dharma/semabs.rtf

Hence, there is not just 1 correct successor possible for the example but 3. But there are also not more successor possible than 3 because additional successors would represent the same difference-structure as the just produced successors. That is, succ(● ■ ●) = (● ■ ● ○) = (● ■ ● ◆).

This fact hints to a remarkable property of morphic patterns, morphograms, that is unknown to the other paradigms of succession and counting, and that also didn't get any proper attention from scientists that are working with 'morphogrammatics': the *retrograde recursivity.*

What does this notional monster mean? It simply sais that a successor of a morphic pattern is defined by the history of its previous successions, and by nothing else.

This is in sharp contrast to all other models of recurrent counting: there, to an existing sequence, string, number, word, any element of the pre-given alphabet might be added freely. Hence, the number of successors depends on the number of elements of the pre-given alphabet and not on the produced sequence.

Hence, morphic patterns are not recurring to a pre-given alphabet but to the history of the just produced patterns.

Stirlingian case

The pattern ● ■ ● is recognized as a Stirlingian constellation. Its successions are:

$$\text{succ}\left(\bullet \blacksquare \bullet\right) = \begin{pmatrix} \bullet \blacksquare \bullet & \bullet \\ \bullet \blacksquare \bullet & \blacksquare \\ \bullet \blacksquare \bullet & \circ \end{pmatrix}$$

The Kindergarten example suggests, correctly for the Leibniz world,

succ(● ■ ●) = (● ■ ● ■),
and is not considering the other cases that are also correct albeit in a Stirling world.

If a child has found a solution in a Stirling world for the pattern [● ■ ●], it has automatically fund a solution for all other patterns of the same differential form.

The thinking of children, discovering the Stirling world are pattern oriented. Hence, the two other patterns [◆ △ ◆] and [■ ‣ ■] proudly presented by the teacher gets solved in one. They just are of the same pattern.

Following the Kindergarten task, it seems reasonable for a Stirling approach to accept two solutions:

an *iterative* and an *accretive* solution.

Hence,

succ(● ■ ●) $= \begin{pmatrix} \text{● ■ ● ■} \\ \text{● ■ ● ○} \end{pmatrix} \begin{pmatrix} \text{iterative} \\ \text{accretive} \end{pmatrix}$

While the repetition of the blue in (● ■ ● ●) could be considered as a wrong solution of the task in a Stirling world. Therefore, for a Leibniz world, there are 2 wrong solutions depending from a Stirling world: ● ■ ● ● and ● ■ ● ○.

A single solution is not yet uncovering the underlying arithmetical rules. Hence, a teacher should go on with the child to find out if there are rules or if the choice is simply arbitrary.

In a difference – oriented notation we get the following elaborations.

● ■ ● $\boxed{?}$	● ■ ● ■	● ■ ● ○	● ■ ● ●
$\vee\ \vee\ \vee$	$\vee\ \vee\ \vee$	$\vee\ \vee\ \vee$	$\vee\ \vee\ \vee$
1. $\nu\ \ \nu\ \ \boxed{?}$	1. $\nu\ \ \nu\ \ \boxed{?} = \nu$	1. $\nu\ \ \nu\ \ \boxed{?} = \nu$	1. $\nu\ \ \nu\ \ \boxed{?} = \varepsilon$
$\vee\ \vee$	$\vee\ \vee$	$\vee\ \vee$	$\vee\ \vee$
2. $\varepsilon\ \ \boxed{?}$	2. $\varepsilon\ \ \boxed{?} = \varepsilon$	2. $\varepsilon\ \ \boxed{?} = \nu$	2. $\varepsilon\ \ \boxed{?} = \nu$
$\vee$	$\vee$	$\vee$	$\vee$
3. $\boxed{?} = \{\nu, \varepsilon\}$	3. $\boxed{?} = \nu$	3. $\boxed{?} = \nu$	3. $\boxed{?} = \varepsilon$

The successor has two possibilities,

equal or non – equal : $\boxed{?} = \{\nu,\ \varepsilon\}$.

For non – equal ν, the result is ● ■ ● ■.

But this is not yet considering

the possibilities of $\boxed{?}$ on level 2 and 3.

If we set on level 2 the choice for $\boxed{?}$ as ε,

we still get the constellation ● ■ ● ■

and force position 3 to $\boxed{?} = \nu$.

If we set on level 2 to $\boxed{?} = \nu$, we get ● ■ ● ○, and force position 3 to $\boxed{?} = \nu$.

Why not the other way round?

Children are getting quickly bored by too much repetition. Why always adding to the right end of the pattern? Why not to the left end? Or even somewhere inside the pattern?

To answer this request, a left-successor shall be introduced.

Left successor

Example

$$\text{right} - \text{succ}(\bullet \blacksquare \bullet) = \begin{pmatrix} \bullet \blacksquare \bullet \bullet \\ \bullet \blacksquare \bullet \blacksquare \\ \bullet \blacksquare \bullet \circ \end{pmatrix}.$$

$$\text{left} - \text{succ}(\bullet \blacksquare \bullet) = \begin{pmatrix} \bullet \bullet \blacksquare \bullet \\ \blacksquare \bullet \blacksquare \bullet \\ \circ \bullet \blacksquare \bullet \end{pmatrix} = \begin{pmatrix} \bullet \bullet \blacksquare \bullet \\ \bullet \blacksquare \bullet \blacksquare \\ \bullet \blacksquare \circ \blacksquare \end{pmatrix}$$

$$\text{succ}(\bullet \blacksquare \bullet) \neq \text{l} - \text{succ}(\bullet \blacksquare \bullet).$$

But

$\text{succ}(\bullet \blacksquare \bullet) = \text{rev}(\text{l} - \text{succ}(\bullet \blacksquare \bullet))$, that is

$\bullet \blacksquare \bullet \bullet = \text{rev}(\bullet \bullet \blacksquare \bullet)$.

Symmetry:

$\bullet \blacksquare \bullet \blacksquare = \text{tnf}(\blacksquare \bullet \blacksquare \bullet)$,

An interesting asymmetry happens for the accretive case:

$\bullet \blacksquare \bullet \circ \neq \text{tnf}(\circ \bullet \blacksquare \bullet) = \bullet \blacksquare \circ \blacksquare$.

The example shows in an exemplary way how the elements of the pattern are changing their identity to preserve the pattern. The elements are in the service to realize the form of the pattern and are not in the defence of their own identity.

Therefore, the two blue kenograms of the example $\bullet \blacksquare \bullet \blacksquare$ are changed to two red kenograms $\bullet \blacksquare \circ \blacksquare$ in the process of the left-succession.HEnce, we get:

left – succ	pattern	right – succ
○●■● $\Longleftarrow$	$\boxed{●■●}$	$\Longrightarrow$ ●■●●

left – succ	pattern	right – succ
●●■●	$\boxed{●■●}$	●■●●
●■●■	–	●■●■
●■○■	–	●■●○

Palindromes

We might observe that the definition of palindromes as demonstrated for the Leibnizian approach, are using left- and right-prolongations.
Example: With alphabet = {●, ■}, ● S ● $\longrightarrow$ ■(● S ●)■, etcetera.

How does it work in the Stirlingian world?

A palindrome in the Stirling world is still a text that reads forwards and backwards the same.

But what has changed dramatically is the concept of reading. Here, what counts is the deep-structure of the text and not its surface appearance symbolized by signs.

What we learned before is the new turn to read the differences between signs instead of the signs. The deep-structure of a pattern is inscribed by the differences between the signs and not by the signs themselves.

Hence, the simple sign-related mechanism of the Leibniz world to define palindromes becomes obsolete.

Without going into details we might try the following production mechanism for Stirlingian palindromes.

More about the formal aspects:
http://memristors.memristics.com/Grammars%20and%20Programs/Grammars%20and%20Programs.pdf

Palindrome grammar – production

Rule1 : $[P] \Rightarrow [w1Pw2]$

Rule2 : $[P] \Rightarrow [w2Pw1]$

Rule3 : $[P] \Rightarrow [w3Pw3]$

Rule4 : $[P] \Rightarrow [w3Pw4]$.

Rule5 : [if length w odd]

$[P] \Rightarrow wM\,[P]\,wM$

Defs

$P = [w] = [w1\ w2]$

$w3 = add(|w1|, 1)$

$w4 = add(|w3|, 1) = add(add(|w2|, 1), 1).$

$wM = middleElement(w)$

Production examples for even palindromes

P: w1≠w2: [w1=●, w2=■]: P = [●,■]
P: w1=w2: [w1=●, w2=●]: P = [●,●].

	rules	results	
P = [●,●]:	w1Pw2	: [●,●,●,●]	; rule1(=rule2)
	w3Pw3	: [■,●,●,■]	; rule3
	w3Pw4	: [■,●,●,○]	; rule4
P = [●,■]:	w1Pw2	: [●,●,■,■]	; rule1
	w2Pw1	: [■,●,■,●]	; rule2
	w3Pw3	: [○,●,■,○]	; rule3
	w3Pw4	: [○,●,■, ◆]	; rule4

Quite obviosly, a pattern like [○,●,■, ◆] doesn't read forwards and backwards the same in a Leibniz world.
But read as a deep-structural pattern of differences is does. Hence it is a Stirlingian palindrome.

Test

The difference-structure of [○,●,■, ◆] is:

v	–	–
v	***v***	–
v	***v***	***v***

This matrix is obviously symmetric. Hence, it reperesents a palindrome.

The same holds for the next example [○,●,■,○]:

v	–	–
v	***v***	–
e	***v***	***v***

.

These are amazing examples to experience and to think about the distinction of *perception* and *cognition* on an extremely elementary level.

The Mersennian case

The pattern [● ■ ●] is recognized as a genuine Mersennian constellation.
Its successions are:

succ(● ■ ●) ={(● ■ ● ■), (● ■ ● ●)}

Succ(● ■ ● ■) = {● ■ ● ■ ●, ● ■ ● ■ ■},
Succ(● ■ ● ●) = {● ■ ● ● ●, ● ■ ● ● ■}.

The Brownian case

The pattern [● ■ ●] is not recognized as a genuine Brownian constellation. The pattern [● ■ ●] is equivalent to its permutations [●● ■] and [■ ● ●].
Hence there is no direct succession for it.
There is a succession for the Brownian standard form *bnf* [● ■ ●] = [● ● ■].
Thus, the child has first to set the pattern into standard normal form for Brownian patterns.
But that would vialate the order as it is presumed from a Leibniz poit of view.

succ(● ● ■) = (●● ■● / ●● ■■).

Comments

For a first glance the Leibnizian and the Mersennian solutions of the task to prolongate the pattern [● ■ ●] are coinciding in the selected iterative results.

Hence, a teacher has no methods at this stage to detect the underlying rationality of the solution presented by a child. It could be as the child would be preferring the Leibniz approach or the not yet known Mersenne approach.

It could be 'correctly' the Leibnizian approach but it could as well be 'incorrectly' the Mersennian approach.

Hence, it is presumed by such an educational approach for a prolongation of a given sequence that the child is accepting as the only correct model of thinking the Leibnizian model.

Another solution, like the accretive solution, is not just judged as wrong but detected by the teacher as symptomatic for a deviant or depraved mind.

Things are getting complicated, hence the elaboration of the matter should be supported by a computer program, an app on a tablet or by a projection from a desktop computer helping the students and the teacher too.

Reversion

rev(● ■ ○ ●) = (● ○ ■ ●) = (● ■ ○ ●)

Palindromes

Is the pattern (● ■ ○ ●) a palindrome?
In other words, is the pattern (● ■ ○ ●) equal its reversion (● ○ ■ ●)?

The first and the last element are equal. But the core elements are different, hence both patterns are different. But this counts for a surface-analysis only. If we take the deep-structure analysis into account, i.e. if we are studying the differences, then it turns out that both patterns have the same difference-structure. Hence, they are morphogrammatically the same.

Therefore, the pattern (● ■ ○ ●) is palindromic.

Test

```
- ENstructureEN[1,2,3,1];
val it = [[],[N],[N,N],[E,N,N]] : EN list list
- ENstructureEN[1,3,2,1];
val it = [[],[N],[N,N],[E,N,N]] : EN list list
```

Matrix comparison

[1,2,3,1]

v	–	–
v	v	–
e	v	v

[1,3,2,1]

v	–	–
v	v	–
e	v	v

Is the pattern [● ■ ● ○ ■ ○] a palindrome?

This pattern is not a palindrome for the classical approach, because the first and the last elements are not identical: ● ≠ ○.

Is it a palindrome for the Brownian or the Mersennian approach?

It is definitely a palindrome for the Stirling approach. It reads forwards and backwards the same. How is that possible? At first, again, the atoms, elements building blocks as entities are not in the focus. Therefore, the comparison has to compare differences and not entities.

The example shows, the first and the last differences are the same. Based on that, the check procedure goes on.

What says the program?

Taken the pattern as string of signs, it is a palindrome. Obviously not a classical palindrome but a palindrome under relabeling. Relabeling simply relabels the numerals of the reversed sequence into a canonical form. If the relabeled sequence is equal the non-reversed sequence, then it is a palindrome.

```
- ispalindrome [1,2,1,3,2,3];
val it = true : bool
```

Checked as what it is, a morphogram that is defined by its differences of ”N” and “E”, the reversion operation is not qualified to give a correct answer.

```
- ENstructureEN [1,2,1,3,2,3];
val it = [[],[N],[E,N],[N,N,N],[N,E,N,N],[N,N,N,E,N]] : EN list list
```

```
- [[],[N],[E,N],[N,N,N],[N,E,N,N],[N,N,N,E,N]] =
rev([[],[N],[E,N],[N,N,N],[N,E,N,N],[N,N,N,E,N]]);
val it = false : bool
```

```
- rev([[],[N],[E,N],[N,N,N],[N,E,N,N],[N,N,N,E,N]]);
val it = [[N,N,N,E,N],[N,E,N,N],[N,N,N],[E,N],[N],[]] : EN list list
```

2.3.6. Palindrome clusters

A nice chance to escape the presupposition of linear order of patterns is given by the matrix presentation of distributed morphograms and palindromes.

$$\begin{pmatrix} [N, N, E] \\ [N, N, E] \\ [E, E, E] \end{pmatrix} \text{corresponds} \begin{pmatrix} [\bullet \blacksquare \blacksquare] \\ [\bullet \blacksquare \blacksquare] \\ [\blacksquare \blacksquare \blacksquare] \end{pmatrix} \text{flattend to} \begin{pmatrix} \bullet \blacksquare \blacksquare \\ \bullet \blacksquare \blacksquare \\ \blacksquare \blacksquare \blacksquare \end{pmatrix}$$

$$\begin{pmatrix} [N, [N, E]] \\ [N, [N, E]] \\ [N, [N, E]] \end{pmatrix} \text{corresponds} \begin{pmatrix} [\bullet \blacksquare \bullet] \\ [\blacksquare \bullet \blacksquare] \\ [\bullet \blacksquare \bullet] \end{pmatrix} \text{flattend to} \begin{pmatrix} \bullet \blacksquare \bullet \\ \blacksquare \bullet \blacksquare \\ \bullet \blacksquare \bullet \end{pmatrix}$$

2.3.7. Bisymmetry of the palindromic matrix

Things has to be elaborated now into a direction that is supporting active thinking in contrast to passive perception.

A child, but the same holds for an adult, can't see the Stirlingian palindrome as such, it has to elaborate its properties.

Again, there is no reason why a child couldn’t get the mechanism with the help of a teacher.

A palindrome reads forwards and backwards the same. Hence it is a symmetric object. But symmetries have not to be put on a line, i.e. on a uni-dimensional linear order. A table does the job too. But it offers an easier approach to find the symmetry. This symmetry is called a *bilateral* symmetry, and handles the situation with the properties of the matrix: the rows and the columns.

Hence, if the matrix is symmetric in respect of its columns and its rows it is the matrix of a palindrome.

Bisymmetry of the matrix

Bisymmetry = rows x columns of the matrix = columns x rows of the inversion of the matrix.

Mathematically, this operation looks not specially simple and kindergarten adequate. But as a concrete game to change the rows and columns it is an elementary experience realized with some sheets or with the help of the App *MorphoGames*.

The naive method deals with the pattern as they are perceived and not with the differences that are recognized. Hence the inversion of the pattern [● ■ ● ○ ■ ○] is the pattern [○ ■ ○ ● ■ ●]. Both are symmetric and the matrix of the differences are equal. Hence the patterns are palindromic.

But, again, with this approach we are not dealing with the differences as our primary objects but with the patterns with their arbitrary elements.

(a) =[● ■ ● ○ ■ ○]   rev(a) = [○ ■ ○ ● ■ ●]

v	–	–	–	–
e	v	–	–	–
v	v	v	–	–
v	**e**	v	v	–
v	v	v	**e**	v

v	–	–	–	–
e	v	–	–	–
v	v	v	–	–
v	**e**	v	v	–
v	v	v	**e**	v

Morphograms are not dealing with the identity of their elements, but with the pattern defined by the differences between the elements only. Therefore we have to apply a different method. This method is focusing on the differences they are notified with the matrix only.

The method is called bilateral symmetry, in short: bisymmetry.

rev – rows

v	v	v	e	v
v	e	v	v	–
v	v	v	–	–
e	v	–	–	–
v	–	–	–	–

rev – cols

v	–	–	–	–
e	v	–	–	–
v	v	v	–	–
v	e	v	v	–
v	v	v	e	v

bisymmetry

v	e	v	v	v
–	v	v	e	v
–	–	v	v	v
–	–	–	v	e
–	–	–	–	v

rev – cols

–	–	–	–	v
–	–	–	v	e
–	–	v	v	v
–	v	v	e	v
v	e	v	v	v

rev – rows

2.4. Distinction and Difference

What distinguishes the difference between distinctions are not distinguished identities, of logical or ontological nature, but kenograms that are different from signs by their distinction of emptiness and location.

In other words, the emphasis on the differential characterization of morphograms by the rejection of semiotic identities has to be involved into a *complementary* play with distinctions that are distinguished form identifiable signs.

Both distinctions, the difference and the distinguished, are localized on the scriptural level of morphogrammatics.

This complementarity is neutral to the classical distinction of atomic signs and the difference between both classical notions.

It might be speculated that the complementarity of *distinction* and *difference* is open to a connection with the difference of *serial* and *parallel*, understood as a complementarity.

Distinctions are conceived step by step, one after the other, thus they shall be labelled 'serial', while differences between distinctions happens at once. And therefore they should be called 'parallel'.

Thus, the emphasis on differences to characterize morphograms, and say morphic palindromes, has not only to be contrasted to semiotics and its identity construct but complemented with an understanding of kenograms as the non-identical units of the differences of morphograms.

This part of the thematization is, in fact, well known, and got an early elaboration, especially by the work at the Biological Computer Lab, Urbana, Ill. in the 1960s about different levels and techniques of abstraction.

This analysis has some consequences for the proposed educational approach.

Also children should be trained to deal with differences, the practical techniques to exercise it, has to involve 'building blocks' like in a chess game.

Also we might abstract from the figure of the chess game and concentrate on the moves and their rules only, the moves have to be done with some figures.

But again, the figures as entities are not in the focus. It doesn't matter how they look. Their attributes are in fact not involved into the game.

With this balanced approach between 'serial' distinctions and 'parallel' differences we are prepared now to study, explore and experience, the intrinsic features, laws and strategies of morphic games.

The aim of those morphic games is not to help children to understand adult math.

Therefore, my interest is not into topics like *"the continuum and the discrete"*, or other number theoretic notions like *"the analog and the digital"*, "*the cardinal and the ordinal*", well studied by Moshe Klein.

> *"The fact that the children haven't yet been exposed to the formal education systems - hence their thought process is free and unblemished - gave us the feeling that the work with them could be utilized in our research.*
>
> *"We came to the conclusion that kindergarten children have a different way of grasping concepts and a different way of thinking than do adults.*
>
> *"While the so-called "adult Mathematical thinking" is based mostly on Logic, children think in a way that is balanced somewhere in-between logic, intuition, emotion and imagination. We called this thought process "Organic Thinking" and tried to characterize it.*
> *After conducting a number of research meetings we were able to understand*

how it is possible to characterize this thinking mathematically."

In contrast to Moshe Klein's project, I'm not intending to solve Hilbert's 6 Problem, or other serious mathematical problems, with the help of the creativity of Kindergarten children. Nor do I have any attempts to teach children the basics of 'adult' math with the help of the medium of morphogrammatics. And quite obviously, I don't believe in the *"free and unblemished"* innocence of the way children are thinking. Nor do I think that Genetics as we know it is determining the basic rules of numbers and grammar.

My emphasis is just to point on possible fundamental *differences* in the general behavior of cognitive actors.

Therefore, this approach is applicable to all kinds of intelligence, human, alien, animal, robot, sane or depraved, handicapped or super-minds, etcetera.

I also don't have any reasons to believe that children are closer to George Spencer-Brown's *"Laws of Form"* and its calculus of distinction than to the identity games of educated adults.

Unfortunately, many children are proud to use binary classification systems and are applying perfectly binary logic, and all kind of disambiguation and 'de-paradoxing' strategies like adults, and have never had the chance to listen to their own mind set.

Even the smallest children are able to parrot the basics of adult math of their parents and kindergarten teacher. And in this, they are not different from our smart robots. Robots are also making their 'parents' proud.

Nevertheless, there are still chances that some categories of thinking, like individual identity and properties, are not yet glued together, and that some pre-logical flexibility in thinking as we know it from personal experiences and from child development psychology, are still accessible for further development in its own rights.

For Piaget's own child it wasn't a logical contradiction to give 2 answers to one question: Where is your daddy? One answer was: High on the ladder in the tree. And simultaneously, the other answer: On a chair in his office. All for the amusement of the surrounding family adults.

But for the academically interested father Piaget it was a baffling answer, and let him to discover, that the identity perception/cognition for entities and the locations of the entities is a result of growing experiences and is in no sense pre-given. They are two independent domains that are interacting together. For adults, this interaction is frozen to a generally accepted result: identities are located.

The question now is: How can we save this capacity to separate fundamental categories and study them separately and in their interaction without denying the child to develop additionally an 'adult' solution and use this kind of gluing categories together as just one possibility next to other conceptualizations and not as an ultimate necessity.

How is a concept of math working that is able to separate the identity of its written signs from the location they appear?

As we know, Piaget was not looking for a different kind of rationality but tried to reconstruct adult thinking along the categories of Immanuel Kant's epistemology.

But it is just the not yet glued distinction of entity and loci that is fundamental for a morphogrammatic paradigm of operativity and rationality.

> ***John Eberts: Jean Piaget and Immanuel Kant: The Concept of the A Priori***
> *"The child comes to know something at a prelinguistic level of development and later comes to know that very same thing at a verbal level. Unfortunately, we tend to encourage verbalization before the child comes to know that of which he speaks. Yet the child's words use the adult lexicon and we allow ourselves to think the child is with his own thoughts when he is merely replying with our words!"*
>
> http://www.philosophos.com/philosophy_article_32.html

The conclusion of John Ebert's observation, that is confirmed by others too, is not that this is just a conceptual confusion by the adults but it is in a strict sense a rejection of the child's own thinking and constitutes therefore a mental abuse of the child.

This form of child abuse is not yet accepted by the authorities as an abuse and is therefore not yet legally treated as a violation of the human rights of children.

Morphogrammatics is by its introduction and definition located on the deep-structural level of the morphosphere, and is therefore pre-logical, pre-linguistic and pre-semiotic, and hence pre-arithmetical too.

This offers a chance to understand different ways of thinking practically and not just in the disguise of an ideology.

2.4.1. Discalculia: A new challenge for teachers and the pharma-industry?

Wolfram Meyerhöfer
Testen, Lernen und Gesellschaft. Zwischen Autonomie und Heteronomie

"Manifeste Orientierung auf Fachsprachlichkeit und latente Zerstörung des Mathematischen, Illusion der Schülernähe als Verblendung, Kalkülorientierung statt mathematischer Bildung, Misslingen der „Vermittlung" von Realem und Mathematischem bei realitätsnahen Aufgaben. Letzteres gründet in der Nichtbeachtung der Authentizität sowohl des Realen als auch des Mathematischen."

http://www.math.uni-potsdam.de/prof/o_didaktik/am/Veroe/Wolfram_Testen_und_Gesellschaft.pdf

SPIEGEL ONLINE: Warum scheitern die Schulen?

Meyerhöfer: Vor allem deshalb, weil lediglich Rechentechniken eingeübt werden. Viele Lehrer behaupten, dass nur die guten Schüler verstehen könnten, warum die Rechenverfahren funktionieren, die Schwachen bräuchten Techniken. Es ist genau umgekehrt: Die schwachen Schüler können nur rechnen lernen, wenn sie verstehen, warum ein Verfahren funktioniert. Für die starken Schüler ist dieses Wissen wiederum ein Bildungssahnehäubchen, das sie aus der Langeweile des Mathematikunterrichts befreien kann.

SPIEGEL ONLINE: Wie entsteht diese Langeweile?

Meyerhöfer: Im Mathematikunterricht langweilen sich alle. Die guten Schüler langweilen sich, weil sie etwas üben müssen, was sie schon können und weil interessante Fragen umschifft werden. Und die schlechten Schüler langweilen sich, weil sie Rechentechniken ohne Verständnis anwenden. Das kann man nur bis zu einer bestimmten Komplexitätsstufe, darum bröckeln peu à peu immer mehr Schüler weg.

http://www.spiegel.de/schulspiegel/wissen/interview-zur-dyskalkulie-rechenschwaeche-ist-eine-erfundene-krankheit-a-928038.html

Overview and information
http://www.uni-bielefeld.de/psychologie/ae/AE09/beratungsstelle/dyskalkulie.html