# — vordenker-archive —


## Rudolf Kaehr

(1942-2016)


### Title

## Graphematics of Multisets

An application of multisets to graphemtical structurations

### Archive-Number / Categories

3_18 / K03, K08

### Publication Date

2012

### Keywords / Topics

- MULTISETS AND GRAPHEMATIC STRUCTURES: Polycontextural modeling of multisets
- GRAPHEMATICS OF MULTISETS: Example of the general trito-set scheme, Trito-structures, Equality for trito-sets, Deutero-structures, Operations on deutero-sets, Proto-structures
- POLYCONTEXTURAL MULTISET THEORY: Identitive disseminations, Bifurcative disseminations, Dissemination of multiset systems, Junctional operations, De Morgan for 3-junctional operations, Reflectional operations, Dissemination of mixed abstract data types, Algebraic properties of disseminated mset-operations
- POSSIBLE APPLICATIONS OF GRAPHEMATICS FOR MEMBRANE COMPUTING
- APPENDIX: ELEMENTS FOR PROGRAMMING TRITO-GRAMMATICS

### Disciplines

Cybernetics, Computer Sciences, Logic and Mathematics, Theory of Science, Semiotics

### Abstract


Some applications of techniques borrowed from multiset theory to elaborate graphematical systems as 'data' structures with the operations of union, sum, difference and polycontextural dissemination of mixed data structures, like set, multiset, list, trito-, deutero- and protograms. The metaphor of 'team' observation for the study of multisets gets a polycontextural explication and application to the team observation of complexions of heterogeneous heterarchic 'data' structures. The elaborations remain on a 'descriptive' formal level.


### Citation Information / How to cite


**Rudolf Kaehr**: "Graphematics of Multisets", www.vordenker.de (Sommer Edition, 2017) J. Paul (Ed.), http://www.vordenker.de/rk/rk_Graphematics-of-Multisets_2012.pdf


### Categories of the RK-Archive

# Graphematics of Multisets

*An application of multiset techniques to graphematical structurations*

**Rudolf Kaehr Dr.phil**@

 ISSN 2041-4358


## Abstract

Some applications of techniques borrowed from multiset theory to elaborate graphematical systems as 'data' structures with the operations of union, sum, difference and polycontextural dissemination of mixed data structures, like set, multiset, list, trito-, deutero- and protograms. The metaphor of 'team' observation for the study of multisets gets a polycontextural explication and application to the team observation of complexions of heterogeneous heterarchic 'data' structures. The elaborations remain on a 'descriptive' formal level. (work in progress, v.0.5)


## 1. Multisets and graphematic structures

### 1.1. Multisets

#### 1.1.1. Summary of multiset approach

Yuncheng Jiang, Description Logics over Multisets
*"A* naive concept *of multiset was formalized by Blizard. It has the following properties:*
*(i) a multiset is a* collection *of elements in which certain elements may occur more than once;*
*(ii) occurrences of a particular element in a multiset are* indistinguishable*;*
*(iii) each occurrence of an element in a multiset contributes to the* cardinality *of the multiset;*
*(iv) the number of* occurrences *of a particular element in a multiset is a* (finite) *positive integer;*
*(v) the number of* distinguishable *(distinct) elements in a multiset need not be finite; and*
*(vi) a multiset is completely* determined *if we know the* elements *that belong to it and the number of times each element belongs to it."*

*"More concretely, a multiset is a collection of objects in which* repetition *of elements is signifcant."*
http://ceur-ws.org/Vol-654/paper1.pdf

In a multiset, repetition is not only relevant but measured by its *multiplicity*.

*"Multisets form a generalization of sets: "identical" elements can occur a finite number of times."*
*"A multiset X is a pair* (X, ρ)*, where X is a set and* ρ *an* equivalence *relation on X . The set X is called the field of the multiset. Elements of X in the same equivalence class will be said to be of the same sort; elements in different equivalence classes will be said to be of different sorts."*
http://obelix.ee.duth.gr/~apostolo/Articles/mset.pdf

Depending on the definition of the *equivalence* relation on X, different classes might be definied with ρ= equivalence relation with μ=multiplicity and λ=locus as:
*sets*, set = (X, ρ=∅, λ=1, μ=1)
*multisets*, mset = (X, ρ, μ, λ=1)
*tritoset,* tset = ((X, ρ, λ, μ)
and *deutero*- and *protosets* and others.

**Epistemological remarks**

In the words of Wilberger:

"Thus the only possible relations between two *mathematical* objects are 1) they are *equal*, or 2) they are *different*.
"This leads to effectively three possible relations between any two physical objects; they are *different*, they are the *same but separate*, or they are *coinciding and identical*."

This corresponds to the Geman distinctions: *Selbigkeit, Gleichheit, Verschiedenheit.*
Or in English: *equal (identical), equivalent (same), different.*

Hence, elements in multisets are equivalent. They occur in different multiplicity as the same at different places in a not ordered context. But they are nevertheless semiotically identical, i.e. *a* at palce i and *a* at place j, i!=j, of a space or a string, are semiotically identical albeit "*same but separate*".
In contrast, elements in tritograms are equivalent despite their semiotical difference.

### 1.1.2. Recalling multisets

"For each a in A the multiplicity (that is, number of occurrences) of a is the number m(a). If a universe U in which the elements of A must live is specified, the definition can be simplified to just a multiplicity function $m_U$ : U -> N from U to the set N = {0, 1, 2, 3, ...} of natural numbers, obtained by extending m to U with values 0 outside." (Multisets, WiKi)

"The set of all mappings ∝: ∪ -> X is denoted by $\cup^X$."

The sum or (arithmetic) addition of A and B, denoted by A + B or A ∪+ B or A ∪ B, is the mset C such that mC (x) = mA (x) + mB (x), for all x.

"For example, if A = $[a, b]_{4,5}$ and B = $[a, b]_{2,3}$ then A − B = $[a, b]_{2,2}$ ⊂ B contradicting the classical laws that (A − B) ∩ B = ∅ and (A − B) ∪ B = A.
Therefore, < p(Y ), ∪, ∩, −, ∅, Y > is only a lattice (Knuth) and **not a boolean algebra.** " (Sing)

The multiplicity function $m_U$ : U -> N from U to the set N = {0, 1, 2, 3, ...} might be involved with the graphematic abstractions, defining different types of graphematic constellations (systems) in terms of multiset terminology, concepts and formalization.

∝: ∪ -> X might be parametrizised towards graphematical abstractions, hence the system of graphematical multiset shall be defined as: graphem($\cup^X$) = $\cup^X{}_{/graphem}$

**Conflict between Calculus of Indication (CI) and Multisets**
If we accept that the CI belongs to the language of *multisets*, as it is supposed by some experts, it turns out that the equally proposed "*boolean algebraic structure*" of the CI that is characterizing the CI, is not holding properly.

Again, it becomes obvious that the CI, even if it belongs to the graphematic scriptures that are defining the languages of multisets, is of such a minimal complexity that its coincidence with boolean structures becomes arbitrary.

Sounds like: "A free Boolean algebra on no elements, namely **2.**"

"For example, if A = $[a, b]_{4,5}$ and B = $[a, b]_{2,3}$ then A − B = $[a, b]_{2,2}$ ⊂ B contradicting the classical laws that (A − B) ∩ B = ∅ and (A − B) ∪ B = A."

For the special case of the CI with A = $[a, b]_{1,1}$ and B = $[a, b]_{1,1}$ then A − B ∩ B = ∅:
($[a, b]_{1,1}$ - $[a, b]_{1,1}$) ∩ $[a,b]_{1,1}$= ∅ and for
(A − B) ∪ B = A:
($[a, b]_{1,1}$ - $[a, b]_{1,1}$) ∪ $[a, b]_{1,1}$ = $[a, b]_{1,1}$.

**Properties**
multiset
multiplicity of objects
cardinality of the multiset
order of objects is irrelevant

**Algebraic properties of msets**

**mset** = $(\text{mset}, \bigcap, \bigcup, \biguplus, \text{com, ass, idem, iden, distr})$

(i) Commutativity :

$A \biguplus B = B \biguplus A,$

$A \bigcup B = B \bigcup A,$

$A \bigcap B = B \bigcap A.$

(ii) Associativity :

$A \biguplus (B \biguplus C) = (A \biguplus B) \biguplus C;$

$A \bigcup (B \bigcup C) = (A \bigcup B) \bigcup C;$

$A \bigcap (B \bigcap C) = (A \bigcap B) \bigcap C.$

(iii) Idempotency :

$A \bigcup A = A;\ A \bigcap A = A;$ but $A \biguplus A \neq A.$

(iv) Identity laws :

$A \bigcup \varnothing = A;$

$A \bigcap \varnothing = \varnothing;$

$A \biguplus \varnothing = A.$

(v) Distributivity :

$A \biguplus (B \bigcup C) = (A \biguplus B) \bigcup (A \biguplus C);$

$A \biguplus B \bigcap C) = (A \biguplus B) \bigcap (A \biguplus C);$

$A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C);$

$A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C).$

Order : $\bigcap \times \subseteq \bigcup \times \subseteq \biguplus \times.$

### 1.1.3. Polycontextural modeling of multisets

Multisets are mappings from U to N, resulting in tupels (U, N). It might be speculated that such a mapping could be represented by contextural mediation between the to distinguished domans U and N. Therefor, this kind of multisets would be represented in the mediated domains of U an N as (U, N). Both domains, U and N, are covered by a contexture, therefore, the mediation (U,N) is represented by a third contexture that is mediating the contextures for U and M.

Following the fact that multisets are answers to two different questions, a modeling in a polycontextural framework is as natural as other modelings too. One question concerns the *set* of elements, the other question is concerned with the *multiplicity* of the elements of the set. Obviously there is a kind of an order between set-theoretic and multiplicity-theoretic topics. It could even be mentioned that the multiplicity-aspect is a reflexion onto the set-aspect of the multiset construction.

On the other hand it could be argued that classical set-theoretic concepts are polycontextural too. But restricted to a mono-contextural understanding where the multiplicity of elements is always just one. This argument holds for the mono-contextural approach to sets and multiplicity too.

"**Remark 1**. *Any ordinary set A is actually a multiset A, χ A , where χ A is its characteristic function."*

There is also another interesting *circularity* to observe. In classical settings, multiplicity in set-theory, based on cardinality, is itself based on sets. Even if the paradoxes of the naive concept of sets are suspended by different axiomatizations, a new paradox emerges: Multiplicity of multisets is based on the cardinality of ordinary sets. Hence, multisets are 'actually' sets of sets. While any ordinary set is "actually a multiset".

A polycontextural thematization and formalization of the topics of multisets is replacing set- or category-theoretic mappings of different domains for a *mediation* of those domains. Mediation additionally opens up highly flexible and complex constellations that are not easily accessible with the concept and formalism of mappings.

As long as both domains or contextures of a *mset* are just separated and are not interacting and are therefore not changing during the process of manipulation, there is no need to introduce more sophisticated concepts and methods to replace or augment the well established static correlations or mappings in the sense of multiset theory. Multiset operations, like insertion, addition, subtraction, etc. are sufficient to realize change in a static context.

If it is reclaimed that msets are more directly respecting real-world and concrete life situations than their counterpart, the abstract sets of extensional set theory, the proposed claims has to be reduced to the fact of another kind of abstract notions. A separation of the two (or more) domains enables flexible concurrent interactions between otherwise stable and unified domains.

*"However like other multiset theories, they are both* two-sorted *theories where the multiplicities are a different type of objects from the multisets they support. This would require separate axioms for multiplicity arithmetic, and in the infinite case it involves piggybacking on a predefined model of cardinal arithmetic (for example [Blizard 3] uses cardinals in a model of ZF set theory)."* (Dang, 2010, p. 48)

A *one-sorted* approach for multiset theory is given in Dang's thesis *"Symmetric sets and graph models of set and multiset theories"*.

*"Therefore we will now propose a one-sorted account of multi- sets, where multiplicities and sets come from the same universe and follow the same axioms. As a result multiplicities are no longer cardinal numbers but sets themselves, with their own internal structures. The natural ordering of multiplicities will be identified with the subset relation, i.e. intuitively we consider x to be less than y as multiplicities if is a proper subset of y."* (Dang, 2010, p. 48)
http://www.dpmms.cam.ac.uk/~tf/dangthesis.pdf

### 1.1.4. Bifunctorial approach to multisets

A version of a deliberation of mappings that is not yet polycontextural might be achieved with the concept and machinery of 2-categories and bifunctoriality between different types of mappings. Here, the mapping of sets and the mapping of arithmetical multiplicity, both generating the mapping of multisets. Hence, multiset mappings are based on set-theoretical mappings as definitions of multisets.

Interchangeability is a strategy to avoid unneccessary conceptual and formal hierarchies. The strategy of Ur-elements is eliminating the type difference between *sets* and *numbers* in favor of an abstract untyped concept prior to sets and numbers.

**Example**

$$A = \left[a, b\right]_{4,5} : \left(U, N\right)$$

$$\left(U, N\right) = \begin{cases} \text{mapping} & : U \longrightarrow N \ , \ \text{set} - \text{theoretical} \\ \text{bifunctoriality} & : U \otimes N \ , \ \text{category} - \text{theoretical} \\ \text{interchangeability} & : U \amalg N \ , \ \text{polycontextural} \end{cases}$$

*mset*: μ: U --> U,  ν: N --> N

bifunctorial: (μ, ν): $\begin{pmatrix} U_1 & U_2 \\ N_1 & N_2 \end{pmatrix}$: $(N_1 \otimes N_2) \circ (U_1 \otimes U_2) = (N_1 \circ U_1) \otimes (N_2 \circ U_2)$.

**Interchangeability of set, multiplicity and multiset**

$$\begin{bmatrix} g_{set} & - & g_{mset} \\ f_{set} & g_{mult} & - \\ - & f_{mult} & f_{mset} \end{bmatrix}:$$

$$\left(\begin{pmatrix} \left(f_{set} \circ^{1.0.0} g_{set}\right) \\ \amalg_{1.2.0} \\ \left(f_{mult} \circ^{0.2.0} g_{mult}\right) \end{pmatrix} \\ \amalg_{1.2.3} \\ \left(f_{mset} \circ^{0.0.3} g_{mset}\right)\right) = \left(\begin{pmatrix} f_{set} \\ \amalg_{1.2.0} \\ f_{mult} \end{pmatrix} \\ \amalg_{1.2.3} \\ f_{mset}\right) \circ_{1.2.3} \left(\begin{pmatrix} g_{set} \\ \amalg_{1.2.0} \\ g_{mult} \end{pmatrix} \\ \amalg_{1.2.3} \\ g_{mset}\right)$$

## 1.2. Indicational structures as multisets

Indicational structures of the calculus of indication, CI, of George Spencer-Brown's *Laws of Form* had been identified mathematically as multisets (Matzka).
This is no secrete. It was also pointed out by Jeffrey James' *Interpretations of Laws of Form* and by others too.
http://www.lawsofform.org/interpretations.html

Supposed there exists an indicational universe, then events occur as partially ordered collections, called multisets. In CI terms that means that the events are commutative or permutative in respect of the number of observed events. Such an indicational space of events then is algebraically defined by commutativity, associativity and idempondency of its primary operation, i.e. concatenation. Distributivity is characterizing concatenation and superposition (encloser) of the CI.

Unfortunatly, no consequences had been drawn from the comparision between multisets and the CI. Therefore, there are no applications of the mathematical methods and results of multiset theory involved with the study of the CI and its possible generalizations.

On the other hand, multiset notions had been studied mathematically from the angle of set-theory and category theory but there seems no attempt to use those insights to motivate a new concept of formal reasoning.

*Our concern in this paper is what the effect on logic will be if we shift from ordinary sets to multisets, i.e. collections which account not only for types but also for tokens of objects.*

*"Under this interpretation of formulas as extensions, a logic Λ contains exactly the syntactic rules of a calculus of extensions forming a certain kind of structure S . We express this by saying that Λ is the logic of S .*

*E.g. classical logic is the logic of boolean fields of sets (i.e., boolean algebras of sets), intuitionistic logic is the logic of pseudo-boolean fields (like the structure of open sets of a topological space), modal logic is the logic of topological boolean fields (that is, boolean fields equipped with a further interior operator), and so on."*
*http://users.auth.gr/tzouvara/Texfiles.htm/multlog.pdf*

Multiset theory is well founded in first order logic (FOL) and classical set theory. Hence, multiset theory is a new branch of mathematics, like fuzzy sets, but is not touching the fundaments of semiotics, logic and arithmetics as such.

In contrast, the ambitions of the indicational calculus, CI, are trying to develop new fundaments for formal and mathematical reasoning on the base of a restricted "multiset" approach.

From the perspective of multisets, it turns out that the CI is a maximally restricted calculus based on minimal multisets. This is in accordance to the fact that the CI is a minimal graphematic system on the level of permutative partitions for m=2.

$$\text{Brown}_{(n,m)} = \binom{n+m-1}{n} = \left(\!\!\binom{n}{k}\!\!\right)$$

### 1.3. Mersenne structures

Mersenne structures are neither sets nor multisets nor strings but tuples with a Mersenne abstraction that is abstracting from the equality of homogeneous tuples. Hence a kind of restricted tuples.

$$\text{Mersenne}_{(n,m)} = m^n - (m-1).$$

## 2. Graphematics of multisets

### 2.1. Graphematics

#### 2.1.1. Little typology of graphematical systems

In contrast to the 3 graphematic systems of semiotics (identity systems, Leibniz), indicational systems (multisets, Brownian) and Mersenne systems, that are all three supporting, in different ways, the semiotic concept of identity of signs, the kenomic systems of graphematics, i.e. the trito-, deutero- and proto-systems, are involved in a subversion of the semiotic principle of identity.

The mentioned 3 graphematic systems had been studied also under the names of Stirling, Pascal and Leibniz systems or scriptural approaches of a general theory of graphematics.

There are at least two strategies to develop more reality-adequate formalisms. One is to involve parametrization over a multitude of “concrete” domains, producing a bulk of specialized ‘data types’. The other approach is to construct an even more abstract formalism to cover structures, like over-determination, interaction, mediation, etc., not accessible to concretized formalisms.

Such new abstractions are tackling with new relationships between types and tokens of semiotic and graphematic objects. The multiset account with *“collections which account not only for types but also for tokens of objects”* shall be continued with a dynamization of the type-token relationship of the sign-usage.

| textemes | |
|---|---|
| multiset | tritogram |
| list | deuterogram |
| set | protogram |
| number | |

$m_U : \text{morph}(U) \longrightarrow_{\text{type}} N :$

| systems | objects | types |
|---|---|---|
| sets | identity | extensionality |
| multisets | identity | multiplicity |
| partitions | identity | permutation |
| Mersenne | identity | homogeneity |
| Stirling | kenomic | trito |
| Pascal | kenomic | deutero |
| Cardinal | kenomic | proto |

**Table of types, examples**

**sets :** $\text{set}(MG) = \{\text{aaaaaaaabbbbbbccc}\} = \{a, b, c\}_{1,1,1}$

**trito – sets :** locus dependent monomorphies :

$\text{morphogram } MG = (\text{aaaabbcccaaaabbbb}) : (1^4\, 2^2\, 3^3\, 1^4\, 2^4)$ : ordered partitions with repetitions

**multisets :** locus independent lexical order, multiplicity and permutation :

$\text{multiset}(MG) = [\text{aaaaaaaabbbbbbccc}] : [1^8, 2^6, 3^3] = [a, b, c]_{8,6,3}$

**deutero – sets** : locus independent free lexical order : partitions without repetitions

$\text{deutero}(MG) = \langle aaaaaaaabbbbbbccc \rangle_{\{8,6,3\}}$

**proto – sets** : locus and order independent collection

$\text{proto}(MG) = < aaaaaaaabbbbbbccc >_{[17:3]}$.

**pomsets** : partially ordered multisets.

"The *pomset* type generalizes sets, bags, lists, trees, and other ordered types, and therefore provides a uniform representation for all these types. Intuitively, a pomset can be viewed as a string with a partial order instead of a total order." (Gumbach, Milo, An algebra of pomsets, 1995)

**lists, strings**: totaly ordered multisets.

"A very special case of partially ordered multisets are the *strings* over a given set of elements. Here the partial ordering is actually total. It is well known that strings are free monoids, meaning that they are freely generated by the signature $\Sigma_{str}$ = <ε, ·> with the following equations:

$$\varepsilon \cdot x = x \quad (1)$$
$$x \cdot \varepsilon = x \quad (2)$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad (3)$$

**ε** denotes the *empty* string and **·** *concatenation* of strings; the equations state that concatenating the empty string to the left or right does not change a string, and that concatenation is associative." (Resnik, Deterministic Pomsets, 1994)

**Multisets**. Another very special case of partially ordered *multisets* are the multisets (sometimes called bags) over a given set of elements. Here the elements are actually completely unordered. Multisets are known to constitute free *commutative* monoids; that is, they are freely generated by the signature $\Sigma_{mul}$ = < ε, ∪+ > with the following equations:

$$\varepsilon \cup + x = x \quad (4)$$
$$(x \cup + y) \cup + z = x \cup + (y \cup + z) \quad (5)$$
$$x \cup + y = y \cup + x \quad (6)$$

ε now denotes the *empty multiset* and ∪+ *multiset addition*; the latter is associative and commutative, whereas adding the empty multiset does not change a multiset." (Resnik, Deterministic Pomsets, 1994)

"A *labelled partially ordered set* or *lposet* over **E** is a triple p = < V , <, l > where
- V is an arbitrary set of *vertices* ;
- < ⊆ V × V is an irreflexive and transitive *ordering relation*;
- l: V -> E is a *labelling function*.

A multiset addition is modelled by disjoint *pomset* union:
$p \cup + q = [V_p \cup V_q , <_p \cup <_q , l_p \cup l_q]$ where again the representatives *p* and *q* should be disjoint." (Resnik, Deterministic Pomsets, 1994)

### 2.1.2. Example of the general tritoset scheme

**Tritoset**[**A**] = [aaaabbcccaaaabbbb]

dec(Tritoset) = ($mg_1$, $mg_2$, $mg_3$) ,
loc(dec(Tritoset)) = ($loc_1(mg_1)$, $loc_2(mg_2)$, $loc_3(mg_3)$, $loc_4(mg_1)$, $loc_5(mg_2)$).
kenom($loc_1(mg_1)$) = [aaaa]
kenom($loc_2(mg_2)$) = [bb]
kenom($loc_3(mg_3)$) = [ccc]
kenom($loc_4(mg_1)$) = [aaaa]
kenom($loc_5(mg_2)$) = [bbbb].

**Tritoset[*A*]**

Tritoset = (loc, mg, ken) consists of loci $loc_i$ and monomorphies $mg_i$ and kenoms (kenograms).
loc(tset[*A*]) = ($loc_1$, $loc_2$, $loc_3$, $loc_4$, $loc_5$)
dec(tset[*A*]) = ($mg_1$, $mg_2$, $mg_3$)
ken($mg_1$) = (aaaa), ken($mg_2$) = (bb), ken($mg_3$) = (ccc).
MG : ken ⟶ mg ⟶ loc.

| tset | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ | $loc_5$ |
|---|---|---|---|---|---|
| dec | $mg_1$ | $mg_2$ | $mg_3$ | $mg_1$ | $mg_2$ |
| ken | a | b | c | a | b |
| | a | b | c | a | b |
| | a | ø | c | a | b |
| | a | ø | ø | a | b |

| tset | | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ | $loc_5$ |
|---|---|---|---|---|---|---|
| | mg 1 | mg1 | - | - | mg1 | - |
| | mg 2 | - | mg2 | - | - | mg 2 |
| | mg 3 | - | - | mg 3 | - | - |

Tritosets are mappings from an alphabet of supporting objects U to N (natural numbers): mset: U --> N defining the kenoms of distributed monomorphies. Loci are the places of different or repeated monomorphies. And monomorphies are containing a number of kenoms as objects: Tritoset : monomorphies --> loci --> kenoms.
Repeated monomorphies might differ in the number of kenoms.

Hence, the example Tritoset[A] = [aaaabbcccaaabbbb] gets a *numerical* notation including the order of the monomorphies and the multiplicity of the monomorphies as the number of the kenoms over the 'support' set [a, b, c], with:

$$\text{Tritoset}[A] = [a, b, c]_{1^4\, 2^2\, 3^3\, 1^4\, 2^4}$$

A final explication has to inscribe the *order* of the loci of the monomorphies in the morphogram (tritoset), 1 -5, hence:

$$\text{Tritoset}[A] = [a, b, c]_{1^4_1\, 2^2_2\, 3^3_3\, 1^4_4\, 2^4_5}$$

For A = $[a, b, c]_{1^4\, 2^2\, 3^3\, 1^4\, 2^4}$, with [a,b,c] as *support* set of kenoms in *trito-normal form* (tnf), the indices $_{1^4\, 2^2\, 3^3\, 1^4\, 2^4}$ as numerical *multiplicity* of the kenoms of the monomorphies and the order of the indices, the *positions* (loci, 1 to 5) of the monomorphies (mg). Because of the implicit order of the indices of the loci, the notation of the positions (loci) might be omitted.

$$\text{support-}[a, b, c]\ \text{occr--}\ \mathbf{1}^4_1\, 2^2_2\, \mathbf{3}^3_3\, 1^4_4\, 2^{4\,-\,\text{multiplicity}}_{5\,-\,\text{locus}}$$

Hence, a Tritoset *tset* is defined as a triple of [occurrence, multiplicity, locus] over a kenomic 'support' set.
While a Multiset *mset* is defined as a tuple [occurrence, multiplicity] over an identitive suppoert set.

| types | set | mset | tset | dset | pset | list |
|---|---|---|---|---|---|---|
| locus | − | − | + | − | − | + |
| multiplicity | − | + | + | + | + | + |
| occurrence | + | + | − | − | − | + |
| order | − | − | + | + | − | + |

**Set-theoretical definitions**

"*Definition*: Let S be a nonempty set. A multi-set M with underlying set S is a set of ordered pairs:

$$M = \{(s_i, n_i) | s_i \in S, n_i \in Z^+\},$$

where $n_i$ is the **multiplicity** of the element $s_i$. A multi-set defined as, or using, a set."
http://mathematics-diary.blogspot.co.uk/2012/03/sets-and-multisets.html

*Multisets* are based on 2 distinctions: *elements*, $s_i$, and the *multiplicity* of the occurrence of elements, $n_i$. Hence: $(s_i, n_i)$.
Not mentioned but accepted is the *identity* presumption of the elements, $s_i \in ID$.
Therefore, the full definitions for multisets is: (elements, multiplicity; identity), i.e.

$$M = \{(s_i, n_i) | s_i \in S, n_i \in Z^+; S \in ID\}.$$

*Tritosets* are based on 3 distinctions: elements, $s_i$, multiplicity, $n_i$, location, $l_i$, in the realm (underlying set) of non-identitive kenograms, i.e., $s_i \in$ KENO.

Therefore, the full "set"-theoretic definition for tritosets is: (elements, multiplicity, location; kenomic), i.e.

$$\mathcal{T} = \{[s_i, n_i, l_i] | s_i \in S, n_i \in \mathcal{Z}^+, l_i \in \mathcal{L}; S \in \text{KENO}\}$$

### 2.1.3. Definition schemes for mset, tset, dset and pset

**Tritoset tset[A]:**

$$\text{Tritoset}[A] = \sum mg_{j=\text{loci}}^{i=\text{multiplicity}}$$

$\sum = [a, b, ..]$: support set $\in$ KENOM,
monomorphy $= mg_{j=\text{locus}}^{i=\text{multiplicity}}$

$$\text{Tcard}(n) = \sum_{k=1}^{n} S2(n, k)$$

**Deuteroset dset[A]:**

$$\text{Deuteroset}[A] = \sum mg_{j=\text{loci}=1}^{i=\text{multiplicity}}$$
$$\text{dset}[A] = \sum \prod([A])$$

$\sum = [a, b, ..]$: support set $\in$ KENOM,
$mg_{j=1}^{i=\text{multiplicity}} = \text{partition} \sum(\prod([A]))$

$$\text{Dcard}(n) = \sum_{k=1}^{n} P(n, k)$$

**Protoset pset[A]:**

$\text{Protoset}[A] = \sum mg_{j=1}^{i=\text{multiplicity}}$

$\text{pset}[A] = (n, k).$

$\sum = [a, b, ..]$: support set $\in$ KENOM,
$\sum(A) = n, \prod(A) = k$

$\text{Pcard}(n) = n$

**Multiset mset[A]:**

$\text{Multiset}[A] = \text{Tritoset} \sum mg_{j=\text{loci}=1}^{i=\text{multiplicity}}$

$\text{mset}[A] = [\sum]_{\text{multiplicity}(\text{occr}(i))}$

with
$\sum = [a, b, ..]$: support set $\in$ ID,
occurrence $V \subseteq \sum =$
$\text{occr}_{j=\text{locus}=0}^{i=\text{multiplicity}} = \text{occr}(i) = m_A(x)$

$\text{Mcard}(n, k) = \binom{k+m-1}{k}$

**Set set[A]:**

$\text{Set}[A] = \text{Tritoset} \sum mg_{j=\text{loci}=1}^{i=1}$

$\text{set}[A] = [\sum]_{\text{occr}(i)}$

with
$\sum = [a, b, ..]$: support set $\in$ ID,
occurrence $V \subseteq \sum =$
$\text{occr}_{j=\text{locus}=0}^{i=\text{multiplicity}=0} = \text{occr}(i) = (x)$

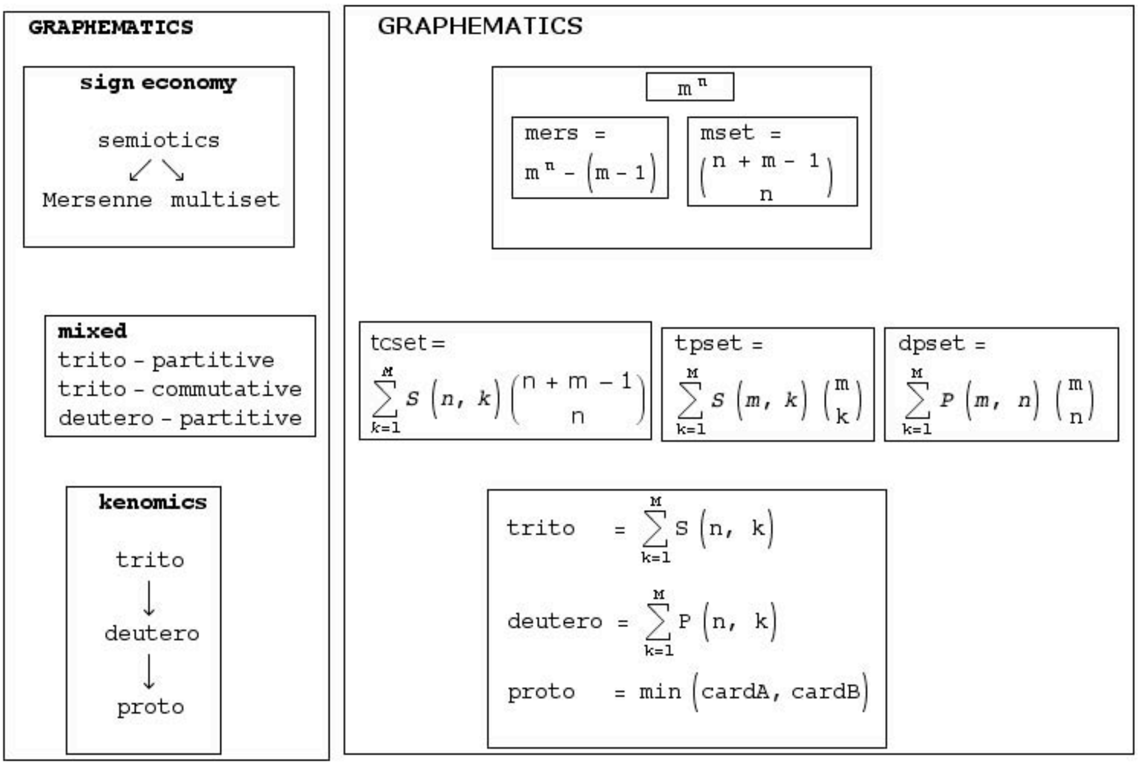
$\text{Scard}(n, m) = m^n$

For A = $[a, b, c]_{1}{}^{4}{}_{2}{}^{2}{}_{3}{}^{3}{}_{1}{}^{4}{}_{2}{}^{4}$, with [a,b,c] as *support* set of kenoms, the indices $_{1}{}^{4}{}_{2}{}^{2}{}_{3}{}^{3}{}_{1}{}^{4}{}_{2}{}^{4}$ as *multiplicity* of the kenoms of the monomorphies and the order of the indices the *positions* (loci, 1 to 5) of the monomorphies (mg). Because of the implicit order of the indices, the notation of the

positions, loci, might be omitted.
In contrast, sets are stripped off of any additional differentiation, i.e. loci=1, multiplicity=1, $\forall x \in D$: $m_A(x)=1$. Example: $[a, b, c]_{1,1,1} = \{a, b, c\}$.

**Multiset**
"The set of distinct elements of an mset is called its *root* or *support*. Formally, the root set of an mset A is the set $\{x | x \in A\}$. The cardinality of the root set of an mset is called its *dimension*." (Sing)

### 2.1.4. Summary




*reduction*: msets --> dpsets --> dsets:

| mset | 1 | 2 | 3 | 4 | num {–} |
|---|---|---|---|---|---|
| 1 | a | a | a | a | $1^4$ |
| 2 | b | b | b | b | $2^4$ |
| 3 | a | a | a | b | $1^3 2^1$ |
| 4 | a | a | b | b | $1^2 2^2$ |
| 5 | a | b | b | b | $1^1 2^3$ |

-->

| dpset | □ | □ | □ | □ | num {–} |
|---|---|---|---|---|---|
| 1 | a | a | a | a | $1^4$ |
| 2 | b | b | b | b | $2^4$ |
| 3 | a | a | a | b | $1^3 2^1$ |
| 4 | a | a | b | b | $1^2 2^2$ |

-->

| dset | □ | □ | □ | □ | num {–} |
|---|---|---|---|---|---|
| 1 | a | a | a | a | $1^4$ |
| 2 | a | a | a | b | $1^3 2^1$ |
| 3 | a | a | b | b | $1^2 2^2$ |

$$\text{mset} = \binom{n+m-1}{n}$$
ident – commutative

$$\text{dpset} = \sum_{k=1}^{M} P(m, n) \binom{m}{n}$$
deutero – partitive

$$\text{dset} = \sum_{k=1}^{M} P(n, k)$$
deutero

*reduction*: tcset --> tpsets --> tsets:

| tcset | 1 | 2 | 3 | 4 | num (–) |
|---|---|---|---|---|---|
| 1 | a | a | a | a | $1^4$ |
| 2 | b | b | b | b | $2^4$ |
| 3 | a | a | a | b | $1^3 2^1$ |
| 4 | a | a | b | a | $1^2 2^2 1^1$ |
| 5 | a | b | a | a | $1^1 2^1 1^2$ |
| 6 | b | a | a | a | $2^1 1^3$ |
| 7 | a | b | b | b | $1^1 2^3$ |
| 8 | b | a | b | b | $2^1 1^1 2^2$ |
| 9 | b | b | a | b | $2^2 1^1 2^1$ |
| 10 | b | b | b | a | $2^3 1^1$ |
| 11 | a | a | b | b | $1^2 2^2$ |
| 12 | a | b | a | b | $1^1 2^1 1^1 2^1$ |
| 13 | a | b | b | a | $1^1 2^2 1^1$ |

-->

| tpset | □ | □ | □ | □ | num(–) |
|---|---|---|---|---|---|
| 1 | a | a | a | a | $1^4$ |
| 2 | b | b | b | b | $2^4$ |
| 3 | a | a | a | b | $1^3 2^1$ |
| 4 | a | a | b | a | $1^2 2^2$ |
| 5 | a | b | a | a | $1^1 2^1 1^2$ |
| 6 | a | a | b | b | $1^2 2^2$ |
| 7 | a | b | a | b | $1^1 2^1 1^1 2^1$ |
| 8 | a | b | b | a | $1^1 2^2 1^1$ |
| 9 | a | b | b | b | $1^1 2^3$ |

-->

| tset | 1 | 2 | 3 | 4 | num (–) |
|---|---|---|---|---|---|
| 1 | a | a | a | a | $1^4$ |
| 2 | a | a | a | b | $1^3 2^1$ |
| 3 | a | a | b | a | $1^2 2^2$ |
| 4 | a | b | a | a | $1^1 2^1 1^2$ |
| 5 | a | a | b | b | $1^2 2^2$ |
| 6 | a | b | a | b | $1^1 2^1 1^1 2^1$ |
| 7 | a | b | b | a | $1^1 2^2 1^1$ |
| 8 | a | b | b | b | $1^1 2^3$ |

$$\text{tcset} = \sum_{k=1}^{M} S(n, k) \binom{n+m-1}{n}$$

trito – commutative

$$\text{tpset} = \sum_{k=1}^{M} S(m, k) \binom{m}{k}$$

trito - partitive

$$\text{tset} = \sum_{k=1}^{M} S(n, k)$$

trito

### 2.1.5. Motivations for sets, strings, multisets, pomsets, tritograms

**Sets, strings, multisets and pomsets**

"Now picture yourself at a bank. You withdraw ten dollars, the teller asks how you want it, you say "*Two fives, please.*" You have thereby specified a one-letter *multiset*. You have not specified an amount, in that you won't settle for ten ones. You have not specified a set, for that would imply particular five-dollar bills. You have however specified a set up to isomorphism, meaning that any two sets of five-dollar bills in bijective correspondence will be equally acceptable. But this is what we mean by a *multiset*. And although you do not appear to have specified an order, this is what we mean by the discrete or empty order, in which no two elements are comparable. Thus you have specified a *pomset* that happens to be a *multiset*.

"Now suppose to this specification we add *"And one at a time, please."* We may distinguish the previous specification from this one as respectively 5|5 and 5;5 or just 55, their *concurrence* versus their *concatenation*. The former is a *multiset*, the latter a *string*, but both are *pomsets*."

"Linearly ordered multisets (labelled chains up to isomorphism) are *strings*. *Pomsets* as partially ordered multisets therefore constitute a generalization of strings to partial orders."
http://homepages.inf.ed.ac.uk/gdp/publications/Teams.pdf

**Trito-sets**

Instead of *"Two fives, please."* it would be enough to ask, independently from the amount, for *"Two of the same, please."* With that, an abstraction from the occurrences of the elements happens. What ever happens, there are just two different possibilities open, both choices are equal or both choices are different. This defines a tritoset but is not yet rich enough to consider the *order* of the unspecified choices.

If there would be 3 choices offered, the possibilities would count up to 5: (aaa), (aab), (aba), (abb) and (abc). The new property is order of unspecified occurrences of elements. The choices (aab) and (aba) are considered as different, while choices (bab) and (aba) are seen as trito-equal.

All together, the abstraction from the occurrences, i.e. the support set, and the rule of order are defining tritosets.
Trito-sets are measured by the Stirling numbers of the second kind: $\sum_{k=1}^{m} \text{Sn2}(m, k)$.
*Example*: trito([aaabbcbb]) = $[a,b,c]_{1^3 2^2 3^1 2^2}$.

In contrast: mset([aaabbcbb]) = $[1^3, 2^4, 3^1]$ = $[a, b, c]_{3, 4, 1}$.

**Deutero-sets**

If we abstract in this model of tritosets from the *order* of the occurrences of the elements, we get

a new class or type of languages, the languages based on *deutero*-sets. Hence, the deutero-sets (aab), (aba), (bba) and (bab), are deutero-equal, equally [aaa] and [bbb], while (aaa) and (aab) are not deutero-equal. Deutero-sets are measured by the sum of partitions: $\sum_{k=1}^{M}$ P(n, m).
*Example*: deutero([aaabbcbb]) = $\{1^3, 2^4, 3^1\}$.

**Proto-sets**
A further abstract that is still keeping some properties of the original pattern is possible with the abstraction of the number of the occurrences of the elements. The proto-set marks the addition of the numbers of separable multiplicities and the number of the occurrence of their elements. Hence, n = Σ multiplicities and m = occurrences, written as ┌m:n┐.
*Example*: proto([aaabbcbb]) = ┌8:3┐, with m=8, n=3.

A reduction to the **cardinality** of a tritogram or a multiset counts the number of the occurrences of elements abstracting from the partition into different kinds of elements, hence proto [8:3] becomes cardinal [3], min{m, n}.

**Order**: multisets => trito => deutero => proto.

### 2.1.6. Formal characterizations

"**Remarks**: Pomsets are only defined up to isomorphism to hide the *identities* of the elements in V, so that only the cardinality of V counts, leaving Σ as the only important set underlying a pomset."
http://www.cs.tau.ac.il/~milo/projects/query_languages/papers/icdt95.ps

"**Definition 1.** A label led partial order or lpo over a set Σ is a structure (V , <=, σ, Σ) where partially orders V and σ : V -> Σ assigns to each element of V an element of Σ.
We think of Σ as an alphabet of actions and V as instances of that alphabet, or events forming a word, with the order of occurrences of letters in the word given by <=."

"**Definition 3**. *A pomset is the isomorphism class of an lpo.*
More intuitively a pomset is an lpo in which we pay no attention to the choice of the set V , other than its cardinality, but retain all other details. Thus if we replace V = {0, 1, 2} by V = {5, 6, 7} without otherwise disturbing either <= or σ the pomset does not change." (Teams p.8)

Does that mean that two pomsets A = (0 1 2) and B = (4 5 6) are equivalent? They are by definition *"defined up to isomorphism"*. But on the level of representations, A and B are not equivalent, A $!=_{pomset}$ B.
In contrast, A and B are trito-equivalent on a "representational" level. Given V = {0, 1, 2} two tritograms A and B over V, with A = (0,1,2,2) and B = (1,2,0,0), are trito-equivalent, A $=_{trito}$ B over their common V.

**Graph representation**
trees, multi-trees, graphs

**List of short definitions**
**1.** A *multi-set* is a collection of objects in which repetition of elements is significant and measured by multiplicity.
2. A *trito-set* is a collection of objects in which identity of objects is irrelevant but distribution and permutation of elements remains significant.
3. A *deutero-set* is a collection of objects in which identity and permutation of objects is irrelevant but partition and repetition of elements remains significant.
4. A *proto-set* is a collection of objects in which just repetition of elements is significant.

### 2.1.7. Combinatoris

"**Sets**, in which the *order* of elements and the *number* of occurrences of each element do not matter.
**Multisets**, in which the *number* of occurrences of each element is important, whereas the *order* of elements does not matter."
items: number of elements Σ, number and order of occurrences of V elements, alphabet and support set.

| types | order | number | elements Σ | identity V | Ex. = $[a, a, b, c, a]$ | combinatorics |
|---|---|---|---|---|---|---|
| posets | □ | □ | □ | □ | □ | □ |
| pomsets | + | + | − | + | □ | □ |
| multisets | − | + | + | + | $[a, b, c]_{3,1,1}$ | $\binom{n+m-1}{n}$ |
| sets | − | − | + | + | $[a, a, b, c, a]_{1,1,1}$ | $m^n$ |
| tsets | + | + | − | − | $(1^2 2^1 3^1 1^1)$ | $\sum_{k=1}^{m} Sn2(m, k)$ |
| dsets | − | + | − | − | $\{1^3 2^1 3^1\}$ | $\sum_{k=1}^{M} P(n, k)$ |
| psets | − | + | + | − | $[5:3]$ | $\min\{m, n\}$ |
| lists | + | + | + | + | $[a, a, b, c, a]_{1,1,2,3,1}$ | $m^n$ |

### *Comparision with the graphematic system*

| Authors | Nummer | Sign Types | Property | Cardinality |
|---|---|---|---|---|
| Gunther, Schadach, von Foerster | IV $(\alpha)$ | Trito – Keno | retrograde recursive, bifunctorial | $\sum_{k=1}^{M} S(n, k)$ |
| Gunther | VI, $(\beta)$ | Deutero – Keno | associative | $\sum_{k=1}^{M} P(n, k)$ |
| Gunther | VIII, $(\gamma)$ | Proto – Keno | commutative | $\min(\text{cardA}, \text{cardB})$ |
| Spencer Brown, Varga, Matzka; Dedekind, de Bruijn | III, $(\delta)$ | Indicational semiotics $(m = 2)$ multisets | commutativ – identitive, associative | $\binom{n+m-1}{n}$ |
| Mersenne | VII $(\varepsilon)$ | Mersenne semiotics | partitive identitive | $M_n = 2^n - 1$ <br> $M_{(m,n)} = m^n - (m - 1)$ |
| Leibniz, Hermes, Schröter | 0 $(i)$ | Semiotics | identitive associative, recursive | $m^n$ |
| Graphematics | II $(\alpha, \epsilon)$ | mixed | trito – partitive semiotics | $\sum_{k=1}^{M} S(n, k)\binom{m}{k}$ |
| Graphematics | I $(\alpha, \delta)$ | mixed | trito – commutative semiotics | $\sum_{k=1}^{M} S(n, k)\binom{n+m-1}{n}$ |
| Graphematics | V $(\beta, \varepsilon)$ | mixed | deutero – partitive semiotics | $\sum_{k=1}^{M} P(n, k)\binom{m}{n}$ |

http://memristors.memristics.com/Graphematics
/Graphematics%20of%20Cellular%20Automata.pdf

**Summary**
There are
1. 3 kenogrammatic systems: *trito*, *proto* and *deutero*.
2. 3 identitive systems: *semiotics*, partition (*Mersenne*), *indication* (Spencer Brown)
3. 3 mixed identical-kenomic systems: *trito-partitive*, *trito-commutative* (trito-Brown), *deutero-partitive.*

### *Number of representations*

**Representations for tritosets**
To deal with abstractions needs representations. A tritogram [abb] is written in normal form and has therefore a number of different representation that are equivalent to the abstract tritogram,

written in normal form.
For the case of just 3 elements involved, the abstract *tritogram* has a representation of 6 concrete realizations, i.e. {[abb], [acc], [baa], [bcc], [caa], [cbb]}, all representing the tritogram [abb].

This number is calculated by the formula :

$$\text{card}[\mu]_{\text{trito}} = \frac{m!}{(m-k)!}$$

Then we get the number 6 for all possible representations for m = 3, k = 2.

**Representations for deutero-sets**

$$\text{card}[\mu]_{\text{deutero}} = \frac{n!}{(1!)^{e_1}(2!)^{e_2}\ldots(n!)^{e_n}}\binom{m}{k}\frac{k!}{e_1!e_2!\ldots e_n!}$$

**Repesentations for proto-sets**

$$\text{card}[\mu]_{\text{proto}} = \sum_{i=1}^{P(n,k)}\frac{n!}{(1!)^{e_{i1}}(2!)^{e_{i2}}\ldots(n!)^{e_{in}}}\binom{m}{k}\frac{k!}{e_{i1}!e_{i2}!\ldots e_{in}!}$$

Schadach gives a full account for all classifications of the graphematic system.
http://www.ballonoffconsulting.com/PDF/1987AppendixII.pdf

**Representations for multisets**
What is the number, card(mset(n, k)), of representations for multisets?

Let S be a multiset that consists of n objects of which
$n_1$ are of type 1 and indistinguishable from each other.
$n_2$ are of type 2 and indistinguishable from each other.
...
$n_k$ are of type k and indistinguishable from each other
and suppose $n_1 + n_2 + \ldots + n_k = n$.

*What is the number of distinct permutations of the n objects in S?*

$$\text{card}[\mu]_{\text{mset}} = \frac{n!}{n_1!n_2!\ldots n_k!}.$$

(CS 171, Lecture Outline, February 03, 2010)

**Example** 1. How many permutations are there of the mset [abccbccbddb]?
**Solution**. We want to find the number of permutations of the multiset
$[A] = [a,b,c,d]1^1 2^4 3^4 4^2 = \{1 \cdot a, 4 \cdot b, 4 \cdot c, 2 \cdot d\}$.
Thus, $n = 11$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $n_4 = 2$. Then number of permutations is given by

$$\frac{n!}{n_1!n_2!\ldots n_k!} = \frac{11!}{1!4!4!2!} = 330.$$

Thus, the mset $[A] = [a,b,c,d]1^1 2^4 3^4 4^2$ has 330 identitive representations. The notation [abccbccbddb] for [A] is therefore a conventional choice and put into mset-normal form notation.

***Monomorphic approach***

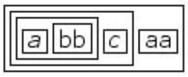
Up to now elements of multisets had been treated as a atomic elements and their occurrences as atomic too. A morphic approach is dealing not with atomic elements but with monomorphies, i.e. with patterns of kenomic elements.

A first approach, albeit still in an identive setting, is given with the definition of multisets with repeated words instead of repeated atomic elements as supports over a common set X.

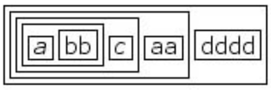
"Usually, a multiset with finite support, M, is presented as a set of pairs {x, M (x)}, for x∈supp(M)."
Paun, et al, DNA Computing, 1998

**Example**
{(ab, 3), (abb, 1), (aa, 2)} = {(ab), (ab), (ab), (abb), (aa), (aa)}

**Application** to tritograms.
**1.** Prolongating the "tail" of a pattern.

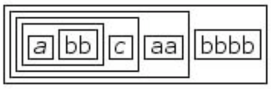
$A =$  =

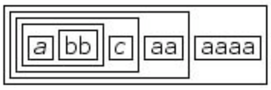
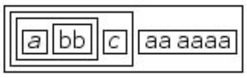
| MG | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ |
|---|---|---|---|---|
| dec | $mg_1$ | $mg_2$ | $mg_3$ | $mg_1$ |
| ken | a | b | c | a |
| | ø | b | ø | a |

$prolong_{(bb,1)_2\,/(dd,2)}(A) =$  =

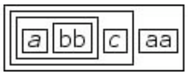
| MG | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ | $loc_5$ |
|---|---|---|---|---|---|
| dec | $mg_1$ | $mg_2$ | $mg_3$ | $mg_1$ | $mg_4$ |
| ken | a | b | c | a | d |
| | ø | b | ø | a | d |
| | ø | ø | ø | ø | d |
| | ø | ø | ø | ø | d |

.

$prolong_{(bb,1)_2\,/(bb,2)}(A) =$ 

$prolong_{(bb,1)_2\,/(aa,2)}(A) =$  = 

A decomposition of A into its monomorphies is given by the table of A with monomorphies $mg_i$ distributed over the loci of their occurrence, $loci_j$.

$A =$  might be written as:

A = $[(a,1)_1, (bb,1)_2, (c,1)_3, (aa,1)_4]$, e.i., [(a,1), (bb,1), (c,1), (aa,1)].

**2.** Another possible prolongation of A might be defined as $prolong_{(bb,\,2)_3}$ where the monomorphy $mg_2$ = (bb) is repeated twice in A in form of all possible trito-occurrences of $(mg_2)_2$ = (bb) at $locus_2$ in A. Therefore, the context or environment of $mg_2$ is determining the occurrences of the monomorphy $mg_2$.

Because the contextual iterations of the monomorphy $mg_2$ are independent in respect to their occurrence, all concrete prolongations might happen at once. Hence, the prolongation is involved with a contextural distribution in the mode of reflection, i.e. iteration into itself, over 4 places and the mediation (∐) of the distributed positions.

$$\text{prolong}_{(bb,\,2)^2}(A) = \left( \begin{array}{l} [(a,\,1),\ (aa,\,2),\ (c,\,1),\ (aa,\,1)]_1 \\ [(a,\,1),\ (bb,\,2),\ (c,\,1),\ (aa,\,1)]_2 \\ [(a,\,1),\ (cc,\,2),\ (c,\,1),\ (aa,\,1)]_3 \\ [(a,\,1),\ (dd,\,2),\ (c,\,1),\ (aa,\,1)]_4 \end{array} \right)$$

for $(bb) =_{trito} (cc) =_{trito} (dd) =_{trito} (aa)$.

The mediation sign $(\amalg)$ between the loci of distribution shall be omitted.

## 2.2. Trito-structures

### 2.2.1. Equality for trito-sets

**Mset-equality A $=_{mset} B$**

**"Equal msets**. Two msets A and B are equal or the same, written as A = B, iff for any object x ∈ D, $m_A(x) = m_B(x)$ or A (x) = B (x). Equivalently, A = B if every element of A is in B and conversely." (Sing)

With emphasis on permutation of the occurrences of the elements:

"Formally, A ∈ $\mathbf{CA}_{n,k}$ is multiset, denoted by A ∈ $\mathbf{MS}_{n,k}$ , if for any permutation π of {1 . . . k}, the local function δA satisfies

$$\forall a_1 , . . . , a_k \in Q_n : \delta A(a1 , . . . , ak ) = \delta A(a_{\pi(1)} , . . . , a_{\pi(k)})."$$

**Example: A $=_{mset} B$**

$[a,b,c]_{1,2,3} = [a,b,c]_{1,2,3}$:

∀ x ∈ D: a,b,c ∈ D

$m_A(x) = m_B(x)$: $m_A(a) = m_B(a) = 1$, $m_A(b) = m_B(b) = 2$, $m_A(c) = m_B(c) = 3$.

**Trito-equality A $=_{trito}$ B based on equivalence**

A $=_{trito}$ B iff for any objects x, y ∈ D, and all loci i,j: $m_A(x_i) = m_B(y_j)$ with 1<=i=j <=|A| and [A] $=_{trito}$[B].

A = $[aabac]_{1\,{}^{2}{}_{2}{}^{1}{}_{1}{}^{1}{}_{3}{}^{1}}$ $=_{trito}$ B = $[bbcba]_{2\,{}^{2}{}_{3}{}^{1}{}_{2}{}^{1}{}_{1}{}^{1}}$

$m_A(x) =_{trito} m_B(y)$: ${}_{1}{}^{2}{}_{2}{}^{1}{}_{1}{}^{1}{}_{3}{}^{1} =_{trito} {}_{2}{}^{2}{}_{3}{}^{1}{}_{2}{}^{1}{}_{1}{}^{1}$

x $=_{trito}$y : [aabac] $=_{trito}$ [bbcba].

**Trito normal form tnf**

A given keno-sequence might not be in a standard normal form (tnf), hence the ML function *tnf*, based on lexical order delivering *kseq* shall be applied.

**Example**

tnf: [cdda] --> [abbc], kseq.

```
- val A = [2, 2, 1, 1] : kseq
> val A = [2, 2, 1, 1] kseq
-tnf A;
>  [1, 1, 2 2] : kseq
```

**Trito-equivalence A $=_{trito}$ B based on the ϵ/v-structure**

Two tritograms [A] and [B] are trito-equivalent iff their ϵ/v-structures are equal.

[A] $=_{trito}$ [B] iff EN([A]) = EN([B]).

**Example**

| ENstructure[A] | | ENstructure[B] |
|---|---|---|
| ν | $=_{trito}$ | ν |
| ν ν | | ν ν |
| ϵ ν ν | | ϵ ν ν |
| [aabc] = [A] | | [▲▲■◇] = [B] |

ENstructure :

datatype EN =E|N;

```
fun delta (i, j) z=
if  (pos i z) = (pos j z)
    then (i, j, E)
     else (i, j, N) ;
```

EN([A]):

```
- ENstructure ["a", "a", "b", "c"];
> [[],
   [(1,2, E),
   [(1, 3, N), (2, 3, N)],
   [(1, 4, N), (2, 4, N), (3, 4, N)]]: enstruct.
```

EN([B]):

```
- ENstructure [" ▲ ", " ▲ ", " ■ " , " ◊ " ];
> [[],
   [(1,2, E),
   [(1, 3, N), (2, 3, N)],
   [(1, 4, N), (2, 4, N), (3, 4, N)]]: enstruct.
```

EN([A]) = EN([B]) <==> [A] $=_{trito}$ [B].

### 2.2.2. Reversion of tritosets

**Reversion for msets**

The reversion of a mset [A] is an unchanged mset. Obviously, the reverse (inverse, dual, reflected) of a mset is obsolete because there is no order of objects that could be changed.

mset[A]:
rev([A]) = [A].

**Example**

mset[A] = [a,b,b] = $[a, b]_{1,2}$
rev([A]) = rev([a,b]) rev(1,2) = $[b,a]_{2,1}$ = $[a,b]_{1,2}$.

**Reversion for tritosets**

Because tritosets are ordered sets, i.e. tritograms, the possibility of reversions for tritosets follows naturally.

tset[A]:
[A] = $[...]_{nnnn}$, with [...]: head, nnnn: multiplicity
rev([A]) = rev([head], multiplicity]) = (rev([head]), rev(multiplicity)])

[A] ∈ sym => rev([A]) $=_{trito}$ [A]
[A] ∉ sym => rev([A]) $!=_{trito}$ [A].

**Example**

[A] ∈ sym, with head, multiplicity∈Sym
[A] = $[aabb]_{1}{}^{2}{}_{2}{}^{2}$
rev([A]) = $[bbaa]_{2}{}^{2}{}_{1}{}^{2}$
tnf(rev([A])) = $[aabb]_{1}{}^{2}{}_{2}{}^{2}$
Hence, [A] $=_{trito}$ rev([A]).

[B] ∈ sym, with head∈Sym, multiplicity∉Sym
[B] = $[aabbb]_{1}{}^{2}{}_{2}{}^{3}$
rev([B]) = rev([aabbb]) $_{rev(1^{2}2^{3})}$
rev([B]) = [bbbaa]) $_{2}{}^{3}{}_{1}{}^{2}$
tnf(rev([B])) = $[aaabb]_{1}{}^{3}{}_{2}{}^{2}$
Hence, [B] $!=_{trito}$ rev([B]).

[C] ∉ sym
C = $[aabac]_{1}{}^{2}{}_{2}{}^{1}{}_{1}{}^{1}{}_{3}{}^{1}$
rev([C]) = $[cabaa]_{3}{}^{1}{}_{1}{}^{1}{}_{2}{}^{1}{}_{1}{}^{2}$
tnf(rev([C])) = $[abcbb]_{1}{}^{1}{}_{2}{}^{1}{}_{3}{}^{1}{}_{2}{}^{2}$
Hence, $[aabac]_{1}{}^{2}{}_{2}{}^{1}{}_{1}{}^{1}{}_{3}{}^{1}$ $!=_{trito}$ $[abcbb]_{1}{}^{1}{}_{2}{}^{1}{}_{3}{}^{1}{}_{2}{}^{2}$.

**Rules**

rev(rev([A])) = [A].

rev([]) $=_t$ [],
rev([a]) $=_t$ [a],
rev([aa]) $=_t$ [aa],
rev([ab]) $=_t$ [ab],
succ([A])) $=_t$ ([A]) $\cup +_t$ [$a_i$], i $\in$ ken([A]) +1
rev(succ([A])) != succ(rev([A])).

**Reflectional morphogrammatics**

Reversions as basic operations for a reflector-based morphogrammatics. (cf. Morphogrammatik, 1993).

A matrix $M^{(3,3)} = \left(\begin{array}{c|c|c|c} M & 1 & 2 & 3 \\ \hline 1 & a & a & c \\ \hline 2 & a & b & b \\ \hline 3 & c & b & c \end{array}\right)$ is easily decomposed into its 2x2-submatrices: $\begin{array}{c|c|c} A & 1 & 2 \\ \hline 1 & a & a \\ \hline 2 & a & b \end{array}$, $\begin{array}{c|c|c} B & 2 & 3 \\ \hline 2 & b & b \\ \hline 3 & b & c \end{array}$, $\begin{array}{c|c|c} C & 1 & 3 \\ \hline 1 & a & c \\ \hline 3 & c & c \end{array}$.

It might sound reasonable to interprete these tritogrammatic patterns, A, B, C, by logical values, resulting into the logical pattern [$\vee \vee \wedge$].

rev([A]) = $\begin{array}{c|c|c} A & 1 & 2 \\ \hline 1 & b & a \\ \hline 2 & a & a \end{array} =_{trito} \begin{array}{c|c|c} A & 1 & 2 \\ \hline 1 & a & b \\ \hline 2 & b & b \end{array}$, logically rev([$\vee$]) = [$\wedge$]. But rev([A]) has to be considered in the context of [ABC], hence

$rev_A$([ABC]) = $\left(\begin{array}{c|c|c|c} [ABC] & 1 & 2 & 3 \\ \hline 1 & b & a & c \\ \hline 2 & a & a & b \\ \hline 3 & c & b & c \end{array}\right) =_{trito} \left(\begin{array}{c|c|c|c} [ABC] & 1 & 2 & 3 \\ \hline 1 & a & b & c \\ \hline 2 & b & b & b \\ \hline 3 & c & b & c \end{array}\right)$, hence logically: $rev_{\vee}[\vee \vee \wedge] = [\wedge \vee \wedge]$.

$rev_{AB}$([ABC]) = $\left(\begin{array}{c|c|c|c} [ABC] & 1 & 2 & 3 \\ \hline 1 & c & b & c \\ \hline 2 & b & b & a \\ \hline 3 & c & a & a \end{array}\right) =_{trito} \left(\begin{array}{c|c|c|c} [ABC] & 1 & 2 & 3 \\ \hline 1 & a & b & c \\ \hline 2 & b & b & c \\ \hline 3 & c & c & c \end{array}\right)$, hence logically: $rev_{\vee \vee}[\vee \vee \wedge] = [\wedge \wedge \wedge]$.

$rev_{ABC}$([ABC]) = $\left(\begin{array}{c|c|c|c} [ABC] & 1 & 2 & 3 \\ \hline 1 & c & a & c \\ \hline 2 & a & b & b \\ \hline 3 & c & b & a \end{array}\right) =_{trito} \left(\begin{array}{c|c|c|c} [ABC] & 1 & 2 & 3 \\ \hline 1 & a & c & a \\ \hline 2 & c & b & b \\ \hline 3 & a & b & c \end{array}\right)$, hence logically: $rev_{\vee \wedge}[\vee \vee \wedge] = [\diamond \vee \vee]$.

http://works.bepress.com/thinkartlab/15/

### 2.2.3. Insertion and prolongation

**Insertion into a multisets**

"The *insertion* of an element x into an mset A gives rise to a new mset A' = A+x such that $m_{A'}(x) = m_A(x)+1$ and $m_{A'}(y) = m_A(y)$ for all x!=y." (Sing)

**Example: multiset insertion**

A = $[a,a,b,b,c]_{2,2,1}$
A' = A + x, for x = a:
$m_{A'}(a) = m_A(a) + 1$: $m_{A'}(a) = 3$
$m_{A'}(y) = m_A(y)$ : for y = b, c, thus
A' = $[a,a,a,b,b,c]_{3,2,1}$.
A' = A + x, for x = b:
$m_{A'}(b) = m_A(b) + 1$: $m_{A'}(b) = 3$
$m_{A'}(y) = m_A(y)$ : for y = a, c, thus
A' = $[a,a,b,b,b,c]_{2,3,1}$.

For two equal multisets [A] and [B], [A] = [B], an insertion of the same element "x" into the msets [A] and [B] restores the equality: [A] = [B] <==> $[A]/x_1 = [B]/x_2$, $x_1 = x_2$. And obviously, for $x_1 \neq x_2$: [A] = [B] <==> $[A]/x_1 \neq [B]/x_2$. Interestingly, this restriction isn't leading for insertions and substitutions in tritosets.

Because multisets are sets with repetition and not ordered sets, the insertion can be applied to each element of the multiset without changing the definition of insertion.
For an ordered set, an insertion might be defined as an addition to the last element and not to any elements inside the order. This is not a necessary restriction but should be applied at first for the examples of morphogrammatic insertions.

**Insertion into morphograms: prolongation**

**Trito-prolongation**

Insertion into multisets has a correspondance to *prologations* in morphogrammatics. Insertion in morphograms of trito, deutero- and proto-structure has different applications. This shall be restricted to the mode of simple *prolongation* in morphograms.

$MG' = MG +_{trito} x$,
$x$ = kenom, $x \in MG+1$
$+_{trito}$ is a retro-grade addition depending on the complexity of MG plus 1.
With the temporary restriction of an addition to the monomorphy of the last locus:
$MG = (mg_1, mg_2, ..., mg_n)$ and
$MG' = (loc_1(mg_1), loc_2(mg_2), loc_3(mg_i), ..., loc_{n-1}(mg_i), loc_n(mg_i+1))$.
For i,j=1,2: $loc_i(mg_j)$, i=j.

MG = [[a] [bb] [c]] =

| MG | $loc_1$ | $loc_2$ | $loc_3$ |
|---|---|---|---|
| Dec | $mg_1$ | $mg_2$ | $mg_3$ |
| Ken | *a* | *b* | *c* |
| | ∅ | *b* | ∅ |

prolongation([[a] [bb] [c]]) : MG $+_i$ x, and tables.

[[a] [bb] [c] [a]] =

| MG | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ |
|---|---|---|---|---|
| Dec | $mg_1$ | $mg_2$ | $mg_3$ | $mg_1$ |
| Ken | *a* | *b* | *c* | *a* |
| | ∅ | *b* | ∅ | ∅ |

[[a] [bb] [c] [b]] =

| MG | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ |
|---|---|---|---|---|
| Dec | $mg_1$ | $mg_2$ | $mg_3$ | $mg_2$ |
| Ken | *a* | *b* | *c* | *b* |
| | ∅ | *b* | ∅ | ∅ |

[[a] [bb] [c] [c]] =

| MG | $loc_1$ | $loc_2$ | $loc_3$ |
|---|---|---|---|
| Dec | $mg_1$ | $mg_2$ | $mg_3$ |
| Ken | *a* | *b* | *c* |
| | ∅ | *b* | *c* |

[[[a] [bb] [c]] [d]] =

| MG | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ |
|---|---|---|---|---|
| Dec | $mg_1$ | $mg_2$ | $mg_3$ | $mg_4$ |
| Ken | *a* | *b* | *c* | *d* |
| | ∅ | *b* | ∅ | ∅ |

**Numerical notation**

num([[a] [bb] [c]]) = [abbc], $[1^1 2^2 3^1]$ = $[a, b, c]_{1^1 2^2 3^1}$

**Prolongation**

$$\text{num}\left(\text{prolong}\left(\boxed{\boxed{\boxed{a}\,\boxed{bb}}\,\boxed{c}}\right)\right) = \begin{pmatrix} [abbca]_{[1^1 2^2 3^1 1^1]} \\ [abbcb]_{[1^1 2^2 3^1 2^1]} \\ [abbcc]_{[1^1 2^2 3^2]} \\ [abbcd]_{[1^1 2^2 3^1 4^1]} \end{pmatrix} = \begin{pmatrix} [a, b, c]_{[1^1 2^2 3^1 1^1]} \\ [a, b, c]_{[1^1 2^2 3^1 2^1]} \\ [a, b, c]_{[1^1 2^2 3^2]} \\ [a, b, c, d]_{[1^1 2^2 3^1 4^1]} \end{pmatrix}.$$

$$\text{prolong}\left(\left[1^1 2^2 3^1\right]\right) = \begin{pmatrix} [1^1 2^2 3^1 1^1] \\ [1^1 2^2 3^1 2^1] \\ [1^1 2^2 3^2] \\ [1^1 2^2 3^1 4^1] \end{pmatrix}.$$

### 2.2.4. Prolongation and induction

**Induction for multisets**

Well – foundedness and the induction principle are well studied for multisets.

$$\frac{P(\emptyset) \bigwedge (\forall a)(\forall M)\left[P(M) \longrightarrow P(M \uplus \{a\})\right]}{(\forall M) P(M)}$$

where the predicate $P(M)$ stands for $M \in W(M(T), <_1)$.

http://www.cs.us.es/~mjoseh/pub/Proving_termination_with_multiset_orderings_in_PVS.pdf

**Induction for tritosets**

Induction for tritosets is not yet studied. But the hints are well established. The succesuccessor operation for the induction is a multiple-successor operation. Therefore, the induction follows in parallel along different branches. In this sense it follows that multset conclusions are, like set- or popositional conclusions, still perceived as single-conclusion systems while tritosets and tritograms are demanding multiple-conclusion systems.

Multiple-conclusion logic might still be in its infancy (D.S. Shoesmith, T.J. Smily, Multiple-Conclusion Logic, 1978) but tritogrammatical systems are offering strong approaches to a further concretization of genuine multiple-conclusion logics.

**Induction scheme for tritosets**

$$\frac{P(\emptyset) \bigwedge (\forall a)(\forall M)\left[P(M) \longrightarrow P(M \uplus_{\text{trito}} \{a\})\right]}{(\forall M) P(M)_1 \mid (\forall M) P(M)_2 \mid \ldots \mid (\forall M) P(M)_n \mid (\forall M) P(M)_{n+1}}$$

The succession range, formalized with (M ∪+ {a}) is depending, not on an abstract atomic element "a" of the set {a} as a successor but on the structure of M that is determining the range of the possible successors.

Hence for [M].ken = ($a_1$, $a_2$, ...,$a_n$), a simple succession model, not yet based on monomorphies, is derived with the mediated parallelism of successions $[A] \uplus a_i \,||\, [A] \uplus a_j$, i!=j:

$$M \uplus_{trito} \{a\} =$$
$$[A] \uplus a_1$$
$$[A] \uplus a_2$$
$$\vdots$$
$$[A] \uplus a_n$$
$$[A] \uplus a_{n+1}.$$

### 2.2.5. Trito-substitution

Substitutions are like concatenation, fusions or merging fundamental concepts for any language. Depending on the definition of the language or scripture, substitutions are involved into interesting interactions. The following gives a short glance into its spectre of differentiations.

**Context rules for tritogrammatic substitution CRS**

$\forall h,\ m_1 \in H_1, m_2 \in H_2,\ m_1 =_{trito} m_2,$
$m_1 \neq_{sem} m_2, h \neq_{sem} m_1,\ m_2,$
$\text{length}(m_1) = \text{length}(m_2),$
$\text{kenom}(m_1) \bigcap \text{kenom}(H_1) = \varnothing,$
$\text{kenom}(m_2) \bigcap \text{kenom}(H_2) = \varnothing :$

$$H_1 =_{trito} H_2$$
$$\Longleftrightarrow$$
$$\text{Subst}_{h/m_1}(H_1) =_{trito} \text{Subst}_{h/m_2}(H_2) \ \ ;\text{modulo CRS}$$

**Trito – substitution example**

Given 2 trito – equivalent tritosets $[A]$ and $[B]$ with $[A] =_{trito} [B]$, which are semiotically different $[A] \neq_{sem} [B]$.

$A$ decomposition of $H_1 = [A]$ and $H_2 = [B]$ delivers :
$\text{Dec}([A]) = ([aa], [bb], [a], [cc]),$
$\text{Dec}([B]) = ([aa], [cc], [a], [bb]),$
The objects $m_1$ and $m_2$ are of equal length, semiotically different but trito – equivalent,
$m_1 = [ddd]$ and $m_2 = [eee]$ shall be inserted at place of $h = [aa]$ of the tritograms $[A]$ and $[B]$.
Conditions of substitution are given with the context rules CRS :
$\text{length}(m_1) = \text{length}(m_2),$
$m_1 \neq_{sem} m_2$ and $m_1 =_{trito} mg_2,\ h \neq_{sem} m_1, m_2$
$\text{sem}(m_i) \cap \text{sem}(H_i) = \varnothing,\ i = 1,\ 2.$
Therefore, the substitution delivers :
$\text{Subst}(\text{Dec}([A]))_{[aa]/[ddd]} = ([ddd], [bb], [a], [cc]),$

$\text{Subst}(\text{Dec}([B]))_{[aa]/[eee]} = ([eee], [cc], [a], [bb]).$

$[A] =_{trito} [B] \Longleftrightarrow \text{Subst}([A])_{[aa]/[ddd]} =_{trito} \text{Subst}([B])_{[aa]/[eee]}.$

$[A'] = [dddbbacc],\ [B'] = [eeeccabb] \Longrightarrow [A'] =_{trito} [B'].$

**Alternative formulation**

**The substitution examples visualized with tables**

Morphograms $MG_1 = \begin{array}{|c|c|} \hline aa & a \\ \hline bb & cc \\ \hline \end{array}$ and $MG_2 = \begin{array}{|c|c|} \hline cc & c \\ \hline bb & dd \\ \hline \end{array}$

| $MG_1$ | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ |
|---|---|---|---|---|
| Dec | $mg_1$ | $mg_2$ | $mg_1$ | $mg_3$ |
| $MG^{1.3}$ | aa | – | a | – |
| $MG^2$ | – | bb | – | – |
| $MG^4$ | – | – | – | cc |

| $MG_2$ | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ |
|---|---|---|---|---|
| Dec | $mg_1$ | $mg_2$ | $mg_1$ | $mg_3$ |
| $MG^{1.3}$ | cc | – | c | – |
| $MG^2$ | – | bb | – | – |
| $MG^4$ | – | – | – | dd |

$$MG_1 =_{MG} MG_2 \Longrightarrow Subst(MG_1)_{[aa]/[ddd]} =_{MG} Subst(MG_2)_{[cc]/[eee]}$$

$Subst(MG_1)_{[aa]/[ddd]} = \begin{array}{|c|c|} \hline ddd & a \\ \hline bb & cc \\ \hline \end{array}$, $Subst(MG_2)_{[cc]/[eee]} = \begin{array}{|c|c|} \hline eee & c \\ \hline bb & dd \\ \hline \end{array}$

| $subst(MG_1)$ | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ |
|---|---|---|---|---|
| Dec | $mg_1$ | $mg_2$ | $mg_1$ | $mg_3$ |
| $MG^{1.3}$ | ddd | – | a | – |
| $MG^2$ | – | bb | – | – |
| $MG^4$ | – | – | – | cc |

| $subst(MG_2)$ | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ |
|---|---|---|---|---|
| Dec | $mg_1$ | $mg_2$ | $mg_1$ | $mg_3$ |
| $MG^{1.3}$ | eee | – | c | – |
| $MG^2$ | – | bb | – | – |
| $MG^4$ | – | – | – | dd |

**Counter – example**

$$MG_1 =_{MG} MG_2 \Longrightarrow Subst(MG_1)_{[aa]/[ccc]} \neq_{MG} Subst(MG_2)_{[cc]/[eee]}$$

| $\begin{array}{\|c\|c\|} \hline ccc & a \\ \hline bb & cc \\ \hline \end{array}$ | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ |
|---|---|---|---|---|
| Dec | $mg_1$ | $mg_2$ | $mg_3$ | $mg_1$ |
| $MG^{1.3}$ | ccc | – | a | – |
| $MG^2$ | – | bb | – | – |
| $MG^4$ | – | – | – | cc |

| $\begin{array}{\|c\|c\|} \hline eee & c \\ \hline bb & dd \\ \hline \end{array}$ | $loc_1$ | $loc_2$ | $loc_3$ | $loc_4$ |
|---|---|---|---|---|
| Dec | $mg_1$ | $mg_2$ | $mg_3$ | $mg_4$ |
| $MG^{1.3}$ | eee | – | c | – |
| $MG^2$ | – | bb | – | – |
| $MG^4$ | – | – | – | dd |

http://memristors.memristics.com/Church-Rosser%20Morphogrammatics/Church-Rosser%20in%20Morphogrammatics.html
http://memristors.memristics.com/MorphoProgramming/Morphogrammatic%20Programming.html
http://memristors.memristics.com/Dominos/Domino%20Approach%20to%20Morphogrammatics.html
http://memristors.memristics.com/semi-Thue/Notes%20on%20semi-Thue%20systems.pdf

### 2.2.6. Trito-sum

$A = [aab]_{[1\ 2\ 2\ 1]}$, with $[a, b]$ as support set : $A = [a, b]_{[1\ 2\ 2\ 1]}$

$B = [bbc]_{[2\ 2\ 3\ 1]}$, with $[b, c]$ as support set : $B = [b, c]_{[2\ 2\ 3\ 1]}$

$sum_{trito}(A, B) = A \uplus_t B.$

$$[aab]_{trito} \uplus_t \begin{pmatrix} \mu_1 B \\ \mu_2 B \\ \mu_3 B \\ \mu_4 B \\ \mu_5 B \\ \mu_6 B \\ \mu_7 B \end{pmatrix} =_t \begin{pmatrix} [aab] \uplus [aab] \\ [aab] \uplus [aac] \\ [aab] \uplus [bba] \\ [aab] \uplus [bbc] \\ [aab] \uplus [cca] \\ [aab] \uplus [ccb] \\ [aab] \uplus [ccd] \end{pmatrix} =_t \begin{pmatrix} [aabaab] \\ [aabaac] \\ [aabbba] \\ [aabbbc] \\ [aabcca] \\ [aabccb] \\ [aabccd] \end{pmatrix}$$

$$\mu(B) = \frac{m!}{(m-k)!} + 1 = \frac{3!}{(3-3)!} + 1 = 7$$

With *root set* notation :

$[a, b]_{1^2\,2^1} \uplus_t [b, c]_{2^2\,3^1} =$

$$[a, b]_{1^2\,2^1} \uplus_t \begin{pmatrix} [a, b]_{1^2\,2^1} \\ [a, c]_{1^2\,3^1} \\ [b, a]_{2^2\,1^1} \\ [b, c]_{2^2\,3^1} \\ [c, a]_{3^2\,1^1} \\ [c, b]_{3^2\,2^1} \\ [c, d]_{3^2\,4^1} \end{pmatrix} =_t \begin{pmatrix} [a, b]_{1^2\,2^1\,1^2\,2^1} \\ [a, b, c]_{1^2\,2^1\,1^2\,3^1} \\ [a, b]_{1^2\,2^3\,1^1} \\ [a, b, c]_{1^2\,2^3\,3^1} \\ [a, b, c]_{1^2\,2^1\,3^1\,1^1} \\ [a, b, c]_{1^2\,2^1\,3^1\,2^1} \\ [a, b, c, d]_{1^2\,2^1\,3^2\,4^1} \end{pmatrix}.$$

**Trito – Non – Commutativity for $\uplus_{trito}$**

$A \uplus_t B \neq B \uplus_t A$

$A = [a, b]_{1^2\,2^1},\ B = [a]_{1^3}$

$[a, b]_{1^2\,2^1} \uplus_t [a]_{1^3} =$

$$[a, b]_{1^2\,2^1} \uplus_t \begin{pmatrix} [a]_{1^3} \\ [b]_{2^3} \\ [c]_{3^3} \end{pmatrix} = \begin{pmatrix} [a, b]_{1^2\,2^1\,1^3} \\ [a, b]_{1^2\,2^4} \\ [a, b, c]_{1^2\,2^1\,3^3} \end{pmatrix}.$$

$[a]_{1^3} \uplus_t [a, b]_{1^2\,2^1} =$

$$[a]_{1^3} \uplus_t \begin{pmatrix} [a, b]_{1^2\,2^1} \\ [b, a]_{2^2\,1^2} \\ [b, c]_{2^2\,3^1} \end{pmatrix} = \begin{pmatrix} [a, b]_{1^5\,2^1} \\ [a, b]_{1^3\,2^2\,1^2} \\ [a, b, c]_{1^3\,2^2\,3^1} \end{pmatrix}.$$

**Mset – commutativity for $\uplus_{mset}$**

$A \uplus_{mset} B = B \uplus_{mset} A$

$A = [a, b]_{2,1}\ ,\ B = [a]_3$

$A \uplus_{mset} B$ :

$[a, b]_{2,1} \uplus_{mset} [a]_3 = [a, b]_{5,1}$

$B \uplus_{mset} A$ :

$[a]_3 \uplus_{mset} [a, b]_{2,1} = [a, b]_{5,1}$

### 2.2.7. Trito-Union

**Multiset union**

"Let A and B be msets. The union of A and B, denoted by A∪B, is the smallest mset C containing both A and B i.e., A ⊆ C and B ⊆ C. In other words, mC(x) = max{mA(x), mB(x)} for all objects x if such a *max* exists; otherwise the *min* is taken which always exists." (Sing)

**Example: trito – union** $_{\text{trito}}(A, B) = A \bigcup_t B.$

$A = [aab]_{1}{}^{2}{}_{2}{}^{1}, \; A = [a, b]_{1}{}^{2}{}_{2}{}^{1}$

$B = [bbc]_{2}{}^{2}{}_{3}{}^{1}, \; B = [b, c]_{2}{}^{2}{}_{3}{}^{1}$

$$\mathbf{1}.\; [aab]_t \bigcup_t \begin{pmatrix} \mu_1 B \\ \mu_2 B \\ \mu_3 B \\ \mu_4 B \\ \mu_5 B \\ \mu_6 B \\ \mu_7 B \end{pmatrix} =_t \begin{pmatrix} [aab] \bigcup [aab] \\ [aab] \bigcup [aac] \\ [aab] \bigcup [bba] \\ [aab] \bigcup [bbc] \\ [aab] \bigcup [cca] \\ [aab] \bigcup [ccb] \\ [aab] \bigcup [ccd] \end{pmatrix} = \begin{pmatrix} [aab] \\ [aabc] \\ [aabba] \\ [aabbc] \\ [aabcca] \\ [aabccb] \\ [aabccd] \end{pmatrix}$$

$$\mathbf{2}.\; [a, b]_{1}{}^{2}{}_{2}{}^{1} \bigcup \begin{pmatrix} [a, b]_{1}{}^{2}{}_{2}{}^{1} \\ [a, c]_{1}{}^{2}{}_{3}{}^{1} \\ [b, a]_{2}{}^{2}{}_{1}{}^{1} \\ [b, c]_{2}{}^{2}{}_{3}{}^{1} \\ [c, a]_{3}{}^{2}{}_{1}{}^{1} \\ [c, b]_{3}{}^{2}{}_{2}{}^{1} \\ [c, d]_{3}{}^{2}{}_{4}{}^{1} \end{pmatrix} = \begin{pmatrix} [a, b]_{1}{}^{2}{}_{2}{}^{1} \bigcup [a, b]_{1}{}^{2}{}_{2}{}^{1} \\ [a, b]_{1}{}^{2}{}_{2}{}^{1} \bigcup [a, c]_{1}{}^{2}{}_{3}{}^{1} \\ [a, b]_{1}{}^{2}{}_{2}{}^{1} \bigcup [b, a]_{2}{}^{2}{}_{1}{}^{1} \\ [a, b]_{1}{}^{2}{}_{2}{}^{1} \bigcup [b, c]_{2}{}^{2}{}_{3}{}^{1} \\ [a, b]_{1}{}^{2}{}_{4}{}^{1} \bigcup [c, a]_{3}{}^{2}{}_{1}{}^{1} \\ [a, b]_{1}{}^{2}{}_{2}{}^{1} \bigcup [c, b]_{3}{}^{2}{}_{2}{}^{1} \\ [a, b]_{1}{}^{2}{}_{2}{}^{1} \bigcup [c, d]_{3}{}^{2}{}_{4}{}^{1} \end{pmatrix} =$$

$$\begin{pmatrix} [a, b]_{1}{}^{2}{}_{2}{}^{1} \\ [a, b, c]_{1}{}^{2}{}_{2}{}^{1}{}_{3}{}^{1} \\ [a, b]_{1}{}^{2}{}_{2}{}^{2}{}_{1}{}^{1} \\ [a, b, c]_{1}{}^{2}{}_{2}{}^{2}{}_{3}{}^{1} \\ [a, b, c]_{1}{}^{2}{}_{2}{}^{1}{}_{3}{}^{2}{}_{1}{}^{1} \\ [a, b, c]_{1}{}^{2}{}_{2}{}^{2}{}_{3}{}^{2}{}_{2}{}^{1} \\ [a, b, c, d]_{1}{}^{2}{}_{2}{}^{1}{}_{3}{}^{2}{}_{4}{}^{1} \end{pmatrix}.$$

**Trito – Non – Commutativity for $\bigcup_{trito}$**

$A \bigcup_t B \neq B \bigcup_t A$

$A = [a, b]_{1^2 2^1},\ B = [a]_{1^3}$

$[a, b]_{1^2 2^1} \bigcup_t [a]_{1^3} =$

$$[a, b]_{1^2 2^1} \bigcup_t \begin{pmatrix} [a]_{1^3} \\ [b]_{2^3} \\ [c]_{3^3} \end{pmatrix} = \begin{pmatrix} [a, b]_{1^2 2^1 1^3} \\ [a, b]_{1^2 2^3} \\ [a, b, c]_{1^2 2^1 3^3} \end{pmatrix}.$$

$[a]_{1^3} \bigcup_t [a, b]_{1^2 2^1} =$

$$[a]_{1^3} \bigcup_t \begin{pmatrix} [a, b]_{1^2 2^1} \\ [b, a]_{2^2 1^2} \\ [b, c]_{2^2 3^1} \end{pmatrix} = \begin{pmatrix} [a, b]_{1^3 2^1} \\ [a, b]_{1^3 2^2 1^2} \\ [a, b, c]_{1^3 2^2 3^1} \end{pmatrix}.$$

## Trito-non-idempotency

$A \bigcup_t A \neq A,$

$A \bigcap_t A \neq A,$

$A \biguplus_t A \neq A.$

**Idempotency for $\bigcup$:**

$A = [a, b]_{1^2 2^1}$

$$[a, b]_{1^2 2^1} \bigcup_t \begin{pmatrix} [a, b]_{1^2 2^1} \\ [b, a]_{2^2 1^1} \\ [b, c]_{2^2 3^1} \end{pmatrix} = \begin{pmatrix} [a, b]_{1^2 2^1} \\ [a, b]_{1^2 2^2 1^1} \\ [a, b, c]_{1^2 2^2 3^1} \end{pmatrix}$$

**Trito - idempotency for $\bigcup$**

$$A \bigcup_t \begin{pmatrix} [A] \\ [A'] \\ [A''] \end{pmatrix} = \begin{pmatrix} [A] \\ [A] + [A'] \\ [A] + [A''] \end{pmatrix}$$

$[A] = [A]_{1^2 2^1}$

$[A'] = [A]_{2^2 1^1},\ [A''] = [A]_{2^2 3^1}$

**Idempotency for $\bigcap$:**

$$[a,b]_{1^2\,2^1} \bigcap_t \begin{pmatrix} [a,b]_{1^2\,2^1} \\ [b,a]_{2^2\,1^1} \\ [b,c]_{2^2\,3^1} \end{pmatrix} = \begin{pmatrix} [a,b]_{1^2\,2^1} \\ [a,b]_{1^2\,2^1\,1^1} \\ [a,b,c]_{1^2\,2^1\,3^1} \end{pmatrix}$$

**Trito - idempotency for $\bigcap$**

$$A \bigcap_t \begin{pmatrix} [A] \\ [A'] \\ [A''] \end{pmatrix} = \begin{pmatrix} [A] \\ [A] + [a] \\ [A] + [c] \end{pmatrix}$$

$$[A] = [A]_{1^2\,2^1}$$

$$[A'] = [A]_{2^2\,1^1},\ [A''] = [A]_{2^2\,3^1}$$

Trito – idempotency for $\uplus$:

$$[a,b]_{1^2\,2^1} \uplus_t \begin{pmatrix} [a,b]_{1^2\,2^1} \\ [b,a]_{2^2\,1^2} \\ [b,c]_{2^2\,3^1} \end{pmatrix} = \begin{pmatrix} [a,b]_{1^4\,2^2} \\ [a,b]_{1^2\,2^3\,1^2} \\ [a,b,c]_{1^2\,2^3\,3^1} \end{pmatrix}$$

**Trito - idempotency for $\uplus$**

$$A \bigcap_t \begin{pmatrix} [A] \\ [A'] \\ [A''] \end{pmatrix} = \begin{pmatrix} [A] \\ [A] \oplus [A'] \\ [A] \oplus [A''] \end{pmatrix}$$

$$[A] = [A]_{1^2\,2^1}$$

$$[A'] = [A]_{2^2\,1^1},\ [A''] = [A]_{2^2\,3^1}$$

### 2.2.8. Trito-Difference

**Multiset difference**

" Difference and complementation. Let A and B be two msets over D and $B \subseteq A$, then $m_{A-B}(x) = m_A(x) - m_{A\cap}$

**Tritoset difference**

$A, B \in \text{TRITO} : \text{difference}_{\text{trito}}(A, B) = A \setminus_t B$

**Example : difference** $A \setminus_{\text{trito}} B$

$A = [a, b, c]_{1^2 2^3 3^2}$ , $B = [a, b]_{1^2 2^3}$ :

$A \setminus_t B$

$$= [a, b, c]_{1^2 2^3 3^2} \setminus_t [a, b]_{1^2 2^1} = [a, b, c]_{1^0 2^2 3^2}$$
$$= [a, b, c]_{1^2 2^3 3^2} \setminus_t [a, c]_{1^2 3^1} = [a, b, c]_{1^0 2^3 3^1}$$
$$= [a, b, c]_{1^2 2^3 3^2} \setminus_t [b, a]_{2^2 1^1} = [a, b, c]_{1^2 2^1 3^2 1^{-1}}$$
$$= [a, b, c]_{1^2 2^3 3^2} \setminus_t [b, c]_{2^2 3^1} = [b, c]_{1^2 2^1 3^1}$$
$$= [a, b, c]_{1^2 2^3 3^2} \setminus_t [c, a]_{3^2 1^1} = [a, b, c]_{1^2 2^3 3^0 1^{-1}}$$
$$= [a, b, c]_{1^2 2^3 3^2} \setminus_t [c, b]_{3^2 2^1} = [a, b, c]_{1^2 2^3 3^0 2^{-1}}$$
$$= [a, b, c]_{1^2 2^3 3^2} \setminus_t [c, d]_{3^2 4^1} = [a, b, c]_{1^2 2^3 3^0 4^{-1}}$$

$$A \setminus_t B = \begin{pmatrix} [a, b, c]_{1^0 2^2 3^2} \\ \hline [a, b, c]_{1^0 2^3 3^1} \\ \hline [a, b, c]_{1^2 2^1 3^2 1^{-1}} \\ \hline [a, b, c]_{1^2 2^1 3^1} \\ \hline [a, b, c]_{1^2 2^3 3^0 1^{-1}} \\ \hline [a, b, c]_{1^2 2^3 3^0 2^{-1}} \\ \hline [a, b, c]_{1^2 2^3 3^0 4^{-1}} \end{pmatrix} = \begin{pmatrix} [a, b]_{1^2 2^2} \\ \hline [a, b]_{1^3 2^1} \\ \hline [a, b, c]_{1^2 2^1 3^2 1^{-1}} \\ \hline [a, b, c]_{1^2 2^1 3^1} \\ \hline [a, b]_{1^2 2^3 1^{-1}} \\ \hline [a, b]_{1^2 2^2} \\ \hline [a, b, c]_{1^2 2^3 3^{-1}} \end{pmatrix}$$

The proposed approach of the example for the difference operation accepts *negative* 'occurrences' of elements. This makes sense only in the *context* of the whole constellation of the trito-grammatic difference operation. Negative occurrences as such, i.e. in isolation, are not (yet) considered.

Normal form: $[a,b,c]_{1^2 2^3 3^0 4^{-1}}$ = [aabbb c=⌀ d=-1] = [aabb d=-1] = [aabbc=-1] = $[a,b,c]_{1^2 2^3 3^{-1}}$

## 2.2.9. Trito-Complement

**Multiset approach**

"Let $\mathfrak{I} = \{A_1, A_2, \ldots\}$ be a family of multisets composed of the elements of the generic set D. Then, the maximum multiset z is defined by $m_Z(x) = \max_{A \in \mathfrak{I}} m_A(x)$ for all $x \in D$ and all $A \in \mathfrak{I}$.

Now, the complement of an mset A, denoted by $\overline{A}$ , is defined as follows:

$\overline{A} = Z - A = \{m_{\overline{A}}(x) . x \mid m_{\overline{A}}(x) = m_Z(x) - m_A(x), \text{ for all } x \in D\}$." (Sing)

"Now define the **difference** $A - B$ between two multisets A and B as $A - B = A + (-B)$ or equivalently by the rule that for any object x $m_{A-B}(x) = m_A(x) - m_B(x)$ .

$(-1) A = - A$

$n A + m A = (n + m) A$, $n (mA) = (nm) A$ .

"We may now derive the 'De Morgan type' laws

$(-A) \cap (-B) = -(A \cup B)$

$(-A) \cup (-B) = -(A \cap B)$

and their relative versions

$(A - B) \cap (A - C) = A - (B \cup C)$

$(A - B) \cup (A - C) = A - (B \cap C)$."

(N J Wildberger, A new look at multisets, 2003)

http://web.maths.unsw.edu.au/~norman/papers/NewMultisets5.pdf

**Example: Trito-complement $\overline{A}$ (- A)**

$A = [a, b, c]_{1}{}^{2}{}_{2}{}^{3}{}_{3}{}^{1}$

$B = [a, b]_{1}{}^{1}{}_{2}{}^{1}$

$$(-1)B = -B = -[a, b]_{1}{}^{1}{}_{2}{}^{1} = [a, b]_{-1}{}^{1}{}_{-2}{}^{1}$$

$$A - B = [a, b, c]_{1}{}^{2}{}_{2}{}^{3}{}_{3}{}^{1} \uplus [a, b]_{-1}{}^{1}{}_{-2}{}^{1} = [a, b, c]_{1}{}^{1}{}_{2}{}^{1}{}_{3}{}^{1}.$$

$$(-A) \cap (-B) = -(A \cup B)$$

$$[a, b, c]_{-1}{}^{2}{}_{-2}{}^{3}{}_{-3}{}^{1} \bigcap [a, b]_{-1}{}^{1}{}_{-2}{}^{1} = -\left([a, b, c]_{1}{}^{2}{}_{2}{}^{3}{}_{3}{}^{1} \bigcup [a, b]_{1}{}^{1}{}_{2}{}^{1}\right)$$

$$[a, b, c]_{-1}{}^{2}{}_{-2}{}^{3}{}_{-3}{}^{1} = -\left([a, b, c]_{1}{}^{2}{}_{2}{}^{3}{}_{3}{}^{1}\right) = [a, b, c]_{-1}{}^{2}{}_{-2}{}^{3}{}_{-3}{}^{1}.$$

$$-((-A) \cap (-B)) = (A \cup B)$$

$$-\left([a, b, c]_{-1}{}^{2}{}_{-2}{}^{3}{}_{-3}{}^{1}\right) = [a, b, c]_{1}{}^{2}{}_{2}{}^{3}{}_{3}{}^{1}.$$

### 2.2.10. Trito-Products (multiplication)

**Multiset multiplication**

"There is also a *multiplicative* operation for multisets. Define A × B, the direct product of the multisets A and B, to be the multiset consisting of all ordered pairs [a, b] with a ∈ A and b ∈ B. By this we mean that $m_{A \times B}$ ([a, b]) = $m_A$ (a) x $m_B$ (b)." (Wildberger)

**Multiset A × B**

For example
[1 2] × [2 3 2] = [[1, 2] [1, 3] [1, 2] [2, 2] [2, 3] [2, 2]].

"A × [] = [] × A = [] .
|A × B| = |A| |B| .
Distributive laws for finite msets
A × (B ∪ C ) = (A × B) ∪ (A × C )
A × (B ∩ C ) = (A × B) ∩ (A × C )
A × (B + C ) = (A × B) + (A × C ) .
Non-commutative and non-associative laws
A × B != B × A
(A × B) × C != A × (B × C ) ." (Wildberger, p.9)

Additionally:
A x [1] = [1] x A = A.

*Trito-multiplication*

Example : trito – multipication A x B

$A = [a, b]_{1}{}^{1}{}_{2}{}^{2}$ , $B = [a, b]_{1}{}^{1}{}_{2}{}^{1}{}_{1}{}^{1}$

kmul([abb], [aba]) =

| kmul | a | b | a | b " | b ' " |
|---|---|---|---|---|---|
| a<br>b<br>b | a<br>b<br>b | b<br>a<br>a | a<br>b<br>b | b<br>c<br>c | c<br>d<br>d |

$$= \begin{pmatrix} [abb'\ baa'\ abb] \\ \hline [abb'\ bcc'\ abb] \\ \hline [abb'\ cdd'\ abb] \end{pmatrix}$$

$$A \times B = mul\left([a, b]_{1}{}^{1}{}_{2}{}^{2}, [a, b]_{1}{}^{1}{}_{2}{}^{1}{}_{1}{}^{1}\right) =$$

$$\begin{pmatrix} [a, b]_{1}{}^{1}{}_{2}{}^{2}{}_{2}{}^{1}{}_{1}{}^{2}{}_{1}{}^{1}{}_{2}{}^{2} \\ \hline [a, b, c]_{1}{}^{1}{}_{2}{}^{2}{}_{2}{}^{1}{}_{3}{}^{2}{}_{1}{}^{1}{}_{2}{}^{2} \\ \hline [a, b, c, d]_{1}{}^{1}{}_{2}{}^{2}{}_{3}{}^{1}{}_{4}{}^{2}{}_{1}{}^{1}{}_{2}{}^{2} \end{pmatrix} = \begin{pmatrix} [a, b]_{1}{}^{1}{}_{2}{}^{3}{}_{1}{}^{3}{}_{2}{}^{2} \\ \hline [a, b, c]_{1}{}^{1}{}_{2}{}^{3}{}_{3}{}^{2}{}_{1}{}^{1}{}_{2}{}^{2} \\ \hline [a, b, c, d]_{1}{}^{1}{}_{2}{}^{2}{}_{3}{}^{1}{}_{4}{}^{2}{}_{1}{}^{1}{}_{2}{}^{2} \end{pmatrix}.$$

**Arithmetic**

$\left|[a, b]_{1^1\, 2^2}\right| = 3$

$\left|[a, b]_{1^1\, 2^1\, 1^1}\right| = 3$

$\left|\text{mul}\left([a, b]_{1^1\, 2^2},\ [a, b]_{1^1\, 2^1\, 1^1}\right)\right| = 9$

Null

$\mathbf{A \times B} = \text{kmul}\left([a, b]_{1^1\, 2^2},\ [a, b]_{1^1\, 2^1\, 1^1}\right) =$

$(a, b): [1^1 \times_{12} (1^1 2^2),\ 2^1 \times_{21} (1^1 2^2),\ 1^1 \times_{12} (1^1 2^2)] =$
$[(1^1 2^2),\ (2^1 1^2),\ (1^1 2^2)] = [1^1 2^3 1^3 2^2].$

$(a, b, c): [1^1 \times_{12} (1^1 2^2),\ 2^1 \times_{23} (1^1 2^2),\ 1^1 \times_{12} (1^1 2^2)] =$
$[(1^1 2^2),\ (2^1 3^2),\ (1^1 2^2)] = [1^1 2^3 3^2 1^1 2^2].$

$(a, b, c, d): [1^1 \times_{12} (1^1 2^2),\ 2^1 \times_{34} (1^1 2^2),\ 1^1 \times_{12} (1^1 2^2)] =$
$[(1^1 2^2),\ (3^1 4^2),\ (1^1 2^2)] = [1^1 2^2 3^1 4^2 1^1 2^2].$

**Table notation of A x B**

$$A \times B = \begin{pmatrix} [a, b]_{1^1 2^2\ 2^1 1^2\ 1^1 2^2} \\ \hline [a, b, c]_{1^1 2^2\ 2^1 3^2\ 1^1 2^2} \\ \hline [a, b, c, d]_{1^1 2^2\ 3^1 4^2\ 1^1 2^2} \end{pmatrix} = \begin{pmatrix} \text{head} & \text{pos1.3} & \text{pos2} \\ \hline [a, b] & [1^1 2^2]^{1.3} & (2^1 1^2) \\ \hline [a, b, c] & [1^1 2^2]^{1.3} & (2^1 3^2) \\ \hline [a, b, c, d] & [1^1 2^2]^{1.3} & (3^1 4^2) \end{pmatrix}$$

Null

$$\text{kmul}\left([A], [B]\right) = [1^1\, 2^2]^{1.3} \begin{bmatrix} \mathbf{2}^1 & \mathbf{3}^1 \\ [1^2] & - \\ [3^2] & [4^2] \end{bmatrix}$$

$A = [a, b]_{1^1\, 2^2}\ ,\ B = [a, b]_{1^1\, 2^1\, 1^1}$

***Trito-multiplication with context rule***

**Context – Rule CRM**

$\forall\, i \in \text{Dec}(MG_{i+1}),\ mg_i \in MG_1,\ mg_i \in MG_2:$

$$\text{kmul}(MG_1,\ MG_2) \text{ iff } \begin{pmatrix} \text{head}(mg)_i \neq \text{head}(mg)_{i+1} \\ \text{body}(mg)_i \neq \text{body}(mg)_{i+1} \end{pmatrix}$$

*Head* = first kenom of *a* monomorphy

*Body* = the rest.

### Decomposition

$\text{kmul}(MG_1,\ MG_2)$ :

$\text{kmul}([1, 2, 2],\ [1, 2, 3, 1]) \Longrightarrow \text{kmul}(([1],\ [2, 2]),\ ([1],\ [2],\ [3],\ [1])).$

$MG_1 = [1, 2, 2] \Longrightarrow [mg_1,\ mg_2]$

$MG_2 = [1, 2, 3,\ 1] \Longrightarrow [mg_1,\ mg_2,\ mg_3,\ mg_1]$

| kmul | $mg_{2.1}$ | $mg_{2.2}$ | $mg_{2.3}$ | $mg_{2.1}$ |
|---|---|---|---|---|
| $mg_{1.1}$<br>$mg_{1.2}$ | $mg_{1.1}$<br>$mg_{1.2}$ | $\{\overline{mg_{1.1}}\}$<br>$\{\overline{mg_{1.2}}\}$ | $\{\overline{\overline{mg_{1.1}}}\}$<br>$\{\overline{\overline{mg_{1.2}}}\}$ | $mg_{1.1}$<br>$mg_{1.2}$ |

$$\begin{pmatrix} mg_{1.1}, \{\overline{mg_{1.1}}\}, \{\overline{\overline{mg_{1.1}}}\}, mg_{1.1} \\ mg_{1.2}, \{\overline{mg_{1.2}}\}, \{\overline{\overline{mg_{1.2}}}\}, mg_{1.2} \end{pmatrix} \in \text{CRM}$$

### Order

$\text{alph}(MG) \longrightarrow \text{num}(MG)$ :

$[abb],\ [abca] \longrightarrow [1, 2, 2],\ [1, 2, 3, 1].$

**Multiplication table for kmul ([1, 2, 2], [1, 2, 3, 1])**

| kmul | pos1 = 1 | pos2 = 2 | pos3 = 3 | pos4 = 1 |
|---|---|---|---|---|
| 1 | **1** | 2 3333 2222 3333 3333 3 | 3 2244 3434 2452 24545 5 | **1** |
| 2 | **2** | 1 1111 3333 4444 4444 4 | 4 3435 1145 1113 53351 6 | **2** |
| 2 | **2** | 1 1111 3333 4444 4444 4 | 4 3435 1145 1113 53351 6 | **2** |

$$\text{kmul}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ 2 \\ \hline\end{array},\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 1 \\ \hline\end{array}\right) = \left\{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 1 \\ \hline 2 & 1 & 4 & 2 \\ 2 & 1 & 4 & 2 \\ \hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1 & 3 & 2 & 1 \\ \hline 2 & 1 & 3 & 2 \\ 2 & 1 & 3 & 2 \\ \hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 1 & 3 & 2 & 1 \\ \hline 2 & 1 & 4 & 2 \\ 2 & 1 & 4 & 2 \\ \hline\end{array},\ \ldots,\ \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 1 \\ \hline 2 & 4 & 6 & 2 \\ 2 & 4 & 6 & 2 \\ \hline\end{array}\right\}$$

**Exclusion table**

| kmul | 1 | 2 ≠ CRM | 3 ≠ CRM | 1 |
|---|---|---|---|---|
| 1 | **1** | 1 1 3 | 1 1 1 1 3 4 5 | 1 |
| 2 | **2** | 2 3 2 | 2 3 4 5 2 2 2 | 2 |
| 2 | **2** | 2 3 2 | 2 3 4 5 2 2 2 | 2 |

**Inclusion table**

$$\begin{bmatrix} 2 & 3 & 4 & 5 & 2 & 2 & 2 & 3 & 4 & 4 & 5 & 5 \\ 1 & 1 & 1 & 1 & 3 & 4 & 5 & 4 & 3 & 5 & 3 & 6 \\ 1 & 1 & 1 & 1 & 3 & 4 & 5 & 4 & 3 & 5 & 3 & 6 \end{bmatrix} \in \text{CRM}$$

**Ordered multiplication table for kmul ([1, 2, 2], [1, 2, 3, 1])**

$$\mathrm{KMUL}^{(3,4)}_{[r_{1.2.2}],[r_{1.2.3.1}]} \quad \Bigg| \quad [r_{1.2.2}]^{1.4} \begin{bmatrix} \mathbf{r_{2.1.1}} & \mathbf{r_{2.3.3}} & \mathbf{r_{3.1.1}} & \mathbf{r_{3.4.4}} \\ r_{3.4.4} & r_{3.1.1} & r_{2.3.3} & r_{2.1.1} \\ \square & r_{3.4.4} & r_{2.4.4} & r_{2.3.3} \\ \square & r_{4.1.1} & r_{4.3.3} & r_{2.5.5} \\ \square & r_{4.5.5} & r_{4.5.5} & r_{4.1.1} \\ \square & \square & \square & r_{4.3.3} \\ \square & \square & \square & r_{4.5.5} \\ \square & \square & \square & r_{5.1.1} \\ \square & \square & \square & r_{5.3.3} \\ \square & \square & \square & r_{5.6.6} \end{bmatrix}$$

**Example**

$[r_{122}\, r_{211}\, r_{233}\, r_{244}\, r_{233}\, r_{122}] \Longrightarrow$

$[a, b, c, d]_{122\ 211\ 233\ 244\ 233\ 122} \Longrightarrow$

$[a, b, c, d]_{1^1 2^2\ 2^1 1^2\ 2^1 3^2\ 2^1 4^2\ 2^1 3^2\ 1^1 2^2} \Longrightarrow$

$[a, b, c, d]_{1^1 2^3 1^2 2^1 3^2 2^1 4^2 2^1 3^2 1^1 2^2}$ .

**Trito – beginnings, recursion head**

$A \times [\,] = [\,] \times A = [\,]$,

$A \times_{iter} [1] = A$,

$[1] \times_{accr} A \neq A$.

For $A = [a]$:

$1 \times_{iter} [a] = [a]$

$1 \times_{accr} [a] = [b]$

For $A = [ab]$:

$1 \times_{iter} [ab] = [ab]$

$1 \times_{accr} [ab] = [bc]$, hence

$$1 \times [ab] = \begin{pmatrix} [ab] \\ [bc] \end{pmatrix}.$$

## 2.3. Deutero-structures

### 2.3.1. Operations on deutero-sets

**Multisets**

**Example**

$A = [a, b]_{4,5}$, $B = [c, d]_{2,3}$, with $[a, b] \neq [c, d]$

**Deutero – sets**

A and B are deutero – multisets, if their objects are deutero – equivalent :

$A = [a, b]_{4,5}$, $B = [c, d]_{2,3}$, with $[a, b] =_d [c, d]$.

**Example : sum** $\biguplus_{deutero}$

$A = [a, b]_{4,5},\ B = [a, b]_{2,3}$

$$A \biguplus_d B = [a, b]_{4,5} \biguplus_d \begin{pmatrix} {}_{\mu 1}B = [a, b]_{2,3} \\ {}_{\mu 2}B = [b, c]_{2.3} \\ {}_{\mu 2}B = [c, d]_{2,3} \end{pmatrix} = \begin{pmatrix} [a, b]_{6,8} \\ [a, b, c]_{4,7,3} \\ [a, b, c, d]_{4,5,2,3} \end{pmatrix}$$

**Example : union** $\bigcup_{deutero}$

$A = [a, b]_{4,5},\ B = [a, b]_{2,3}$

$$A \bigcup_d B = [a, b]_{4,5} \bigcup_d \begin{pmatrix} {}_{\mu 1}B = [a, b]_{2,3} \\ {}_{\mu 2}B = [b, c]_{2.3} \\ {}_{\mu 2}B = [c, d]_{2,3} \end{pmatrix} = \begin{pmatrix} [a, b]_{4,5} \\ [a, b, c]_{4,5,3} \\ [a, b, c, d]_{4,5,2,3} \end{pmatrix}$$

**Example: multiset difference** $\backslash_{mset}$

"Let A and B be two msets, and B ⊆ A. The (arithmetic) difference of B from A, denoted by A \ B or A − B, is the mset C ⊆ A such that mC(x) = mA(x) − mB(x), for all objects x.
In general, mC(x) = mA(x) − m(A ∩B) (x) = max{mA (x) − mB (x), 0}, for all objects x." (Sing)

**Difference** $\backslash_{mset}$

$A = [a, b]_{4,5},\ B = [a, b]_{2,3},$

$A \setminus B = [a, b]_{4,5} \setminus [a, b]_{2,3} = [a, b]_{2,2}.$

**Example : deutero − difference** $\backslash_d$

$A = [a, b, c]_{4,5,4},\ B = [b, c]_{2,3}$

$$\begin{aligned} A \backslash_d B &= [a, b, c]_{4,5,4} \setminus [a, b]_{2,3} = [a, b, c]_{2,2,4} \\ &= [a, b, c]_{4,5,4} \setminus [a, c]_{2,3} = [a, b, c]_{2,5,1} \\ &= [a, b, c]_{4,5,4} \setminus [b, c]_{2,3} = [a, b, c]_{4,3,1} \\ &= [a, b, c]_{4,5,4} \setminus [c, d]_{2,3} = [a, b, c, d]_{4,5,2,-3}. \end{aligned}$$

$$A \backslash_d B = \begin{pmatrix} [a, b, c]_{4,3,1} \\ \hline [a, b, c]_{2,2,1} \\ \hline [a, b, c]_{3,5,1} \\ \hline [a, b, c, d]_{4,5,2,-3} \end{pmatrix}$$

## 2.4. Proto-structures

**Proto − addition**

$(n, k) +_{proto} (m, j) = \{(n + m, k + j - 1), (n + m, k + j)\}.$

**Example : addition** $_{proto}$

$A = [a, b]_{4,5} = [9 : 2],\ B = [a, b]_{2,3} = [5 : 2]:$

$$A +_{proto} B = [9 : 2] +_{proto} [5 : 2] = \begin{pmatrix} [14 : 3] \\ [14 : 4] \end{pmatrix}$$

**Algebraic properties of** $+_{proto}$

$+_{proto} \in \{\text{distributive, commutative, associative}\}$

# 3. Polycontextural multiset theory

## 3.1. Sketch of polycontexturality

### 3.1.1. General framework

"A mapping ∝: ∪ -> X, where ∪ is a universal set and X is a numericset, is called a set if X = {0, 1}; a multiset if X = N, the set of natural numbers with 0; a signed multiset if X = Z, the set of integers." (Sing)

A logical valuation of ∝ for a classical two-valued logic gives: val (∝) --> {t, f}. This corresponds to a *mono*-contextural constellation. For a *poly*-contextural constellation, the mapping has to consider its dissemination in the contextural grid.

In a *polycontextural* setting, the mapping α is disseminated over a *grid* of contextures, producing a framework for disseminated sets and multisets.

Polycontextural valuation for m=3: $val^{(3)}(\propto^{(3)})$ --> $\{t, f\}^{(3)}$:

$$\left(val^1(\propto) \longrightarrow \{t, f\} \amalg_{1.2} val^2(\propto) \longrightarrow \{t, f\}\right) \amalg_{1.2.3} val^3(\propto) \longrightarrow \{t, f\} \text{ with } t_1 \equiv t_3,\ f_1 \cong t_2,\ f_2 \equiv f_3.$$

$$(t, f)^{(3)} = \begin{pmatrix} t_1 \longrightarrow f_1 \amalg t_2 \longrightarrow f_2 \\ \mid \qquad\qquad\qquad\qquad \mid \\ t_3 \longrightarrow\longrightarrow\longrightarrow f_3 \end{pmatrix}.$$

$$val^{(3)}\left(\propto^{(3)}\right) \longrightarrow \{t, f\}^{(3)} = \left(\begin{pmatrix} val(\propto) \longrightarrow \{t, f\}^{1.1} \\ \amalg_{1.2} \\ val(\propto) \longrightarrow \{t, f\}^{2.2} \end{pmatrix} \begin{matrix} \\ \\ \\ \amalg_{1.2.3} \\ val(\propto) \longrightarrow \{t, f\}^{3.3} \end{matrix}\right)$$

$$\left(\begin{array}{c|c|c} val(\propto) \longrightarrow \{t, f\}^{1.1} & - & - \\ \hline - & val(\propto) \longrightarrow \{t, f\}^{2.2} & - \\ \hline - & - & val(\propto) \longrightarrow \{t, f\}^{3.3} \end{array}\right).$$

**Dissemination**

There are at least 5 main types of actions on disseminated multisets or modi of interaction in polycontextural systems to be studied:

1. identitive : **id**: α --> α
2. permutative : **perm**: $α_1α_2$ --> $α_2α_1$
3. reductive : **red**: $α_1α_2$ --> $α_1α_1$
4. reflective (reflectional) : **refl**$_1$: $α_1α_{2}α_{3}$ --> $α_{1.1}α_{1.2}α_{1.3}α_{2.2}$
5. bifurcative (transpositional, interactional) **bif**$_1$: $α_1α_{2}α_{3}$ --> $α_{1.1}α_{2.1}α_{3.1}α_{2.2}α_{3.3}$

http://crossbars.memristics.com/Poly-Crossbars/Poly-Crossbars.pdf

### 3.1.2. Elements of a polycontextural quantification-theory

Multisets are first of all sets, i.e. special sets. Hence, a polycontextural framework for multisets has to consider some elements of a polycontextural set-theory. Further on, set-theory is based on first-order logic with its operators for "all" and "there exist": ∀x (Px) and ∃x (Px).
In a two-valued setting, the tableaux for ∀x and ∃x are well defined. Hence, a distribution of the tableaux rules for quantification over different contextures follows quite naturally.

Another question is, how to define transjunctional quantifiction, i.e. quantification over discontextural universes of logics? Obviouslly, their elements are at once in at least two different contextures or discontextural domains. Therefore, they have to be taken into account as tupels of 2 elements, say *a* and *b*. The variable x of the quantification gets specialized by "a" and "b". Syntactically, the variables are split to run over different contextures. In the case of $Q_1$, $(a^1, b^1)$

$\in U^1$, with $U^1$ as the universe $U^1$ of logic $Log^1$ in $Log^{(3)}$. Because of the tediuos complication of full formalisation, simplifications should support reading and understanding.

**Quantification schemes**

**Scheme for junctional quantification**

$$(id, id, id) : [\exists \forall]_{type,} [\exists \forall]_{type,} [\exists \forall]_{type} :$$

$$\left(\left(x^{(3,3)}\right) P^{(3,3)} \left(x^{(3,3)}\right)\right) \longrightarrow \left(\begin{array}{c|c|c} [\exists \forall]_{1.1} & - & - \\ \hline - & [\exists \forall]_{2.2} & - \\ \hline - & - & [\exists \forall]_{3.3} \end{array}\right).$$

$$Quant = \{\forall, \exists\}, [\exists \forall]$$

$$x^{i.j} \in U^{i.j}$$

$$type = \{set, mset, tritogram, ...\}.$$

**Scheme for transjunctional quantification**:

**Example:** $\left[\underset{type,type,type}{\forall \quad \exists \quad \forall}\right]$

$$(bif, id, id) : \left[\underset{type,type,type}{\forall \quad \exists \quad \forall}\right] \longrightarrow \left[\underset{type,type,type}{\forall \quad \exists \quad \forall}\right] :$$

$$bif_1 : \left[\underset{type,type,type}{\forall \quad \exists \quad \forall}\right] \longrightarrow \forall_{1.1} \mid \exists_{2.1} \mid \forall_{3.1} : type$$

$$id_2 : \exists \longrightarrow \exists : type$$

$$id_3 : \forall \longrightarrow \forall : type$$

$$\text{Scheme for } \left[\underset{type,\, type,\, type}{\forall \quad \exists \quad \forall}\right] : \left(\begin{array}{c|c|c} \forall_{1.1} & \exists_{2.1} & \forall_{3.1} \\ \hline - & \exists_{2.2} & - \\ \hline - & - & \forall_{3.3} \end{array}\right).$$

$$[bif, id, id] : \left[\underset{type,type,type}{\forall \quad \exists \quad \forall}\right]^{(3,3)} P^{(3,3)}\left(x^{(3,3)}\right),$$

$$type < 1.1., 2.1., 3.1; \; 2.2, 3.3 > :$$

$$\left(\forall^{1.1}\left(x^{1.1}\right)\left(P^{1.1}\left(x^{1.1}\right)\right) \diamond \exists^{2.1}\left(x^{2.1}\right)\left(P^{2.1}\left(x^{2.1}\right)\right) \diamond \forall^{3.1}\left(x^{3.1}\right) P^{3.1}\left(x^{3.1}\right)\right.$$

$$\amalg$$

$$\left(\exists^{2.2}\left(x^{2.2}\right) P^{2.2}\left(x^{2.2}\right)\right)$$

$$\amalg$$

$$\left(\forall^{3.3}\left(x^{3.3}\right) P^{3.3}\left(x^{3.3}\right)\right)$$

$$(bif, id, id) :$$

$$\left[\underset{type,type,type}{\forall \quad \exists \quad \forall}\right] : \left(P^{(3,3)}\left(x^{(3,3)}\right)\right) \longrightarrow \left(\begin{array}{c|c|c} \forall(x)P(x) & \exists(x)P(x) & \forall(x)P(x) \\ \hline - & \exists(x)P(x) & - \\ \hline - & - & \forall(x)P(x) \end{array}\right)$$

**Junctional quantifiers**

$$\frac{T_1\ \exists\forall\exists\ x^{(3)}\ P^{(3)}\ x^{(3)}}{T_1 P^{x\,1}_{a\,1}} \qquad \frac{F_1\ \exists\forall\exists\ x^{(3)}\ P^{(3)}\ x^{(3)}}{F_1 P^{x\,1}_{a\,1}}$$

$$\frac{T_2\ \exists\forall\exists\ x^{(3)}\ P^{(3)}\ x^{(3)}}{T_2 P^{x\,2}_{a\,2}} \qquad \frac{F_2\ \exists\forall\exists\ x^{(3)}\ P^{(3)}\ x^{(3)}}{F_2 P^{x\,2}_{a\,2}}$$

$$\frac{T_3\ \exists\forall\exists\ x^{(3)}\ P^{(3)}\ x^{(3)}}{T_3 P^{x\,3}_{a\,3}} \qquad \frac{F_3\ \exists\forall\exists\ x^{(3)}\ P^{(3)}\ x^{(3)}}{F_3 P^{x\,3}_{a\,3}}$$

$$\begin{pmatrix} \frac{\pi:\ \exists(x)\ P(x)}{T_1 P^{x\,1}_{a\,1}\ \big|\ F_1 P^{x\,1}_{a\,1}} & - & - \\ - & \frac{\pi:\ \forall(x)\ P(x)}{T_2 P^{x\,2}_{a\,2}\ \big|\ F_2 P^{x\,2}_{a\,2}} & - \\ - & - & \frac{\pi:\ \exists(x)\ P(x)}{T_3 P^{x\,3}_{a\,3}\ \big|\ F_3 P^{x\,3}_{a\,3}} \end{pmatrix}.$$

**Transjunctional quantifiers**

$$\frac{T_1 Q\exists\exists\ x^{(3)}P^{(3)}x^{(3)}}{\begin{matrix} T_1 P^{x\,1}_{a\,1} \\ T_1 P^{x\,1}_{b\,1} \end{matrix}} \qquad \frac{F_1 Q\exists\exists\ x^{(3)}P^{(3)}x^{(3)}}{\begin{matrix} F_1 P^{x\,1}_{a\,1} \\ F_1 P^{x\,1}_{b\,1} \end{matrix}}$$

$$\frac{T_2 Q\exists\exists\ x^{(3)}P^{(3)}x^{(3)}}{\begin{array}{c|c} T_2 P^{x\,1}_{a\,1} & F_1 P^{x\,1}_{a\,1} \\ T_2 P^{x\,1}_{b\,1} & F_1 P^{x\,1}_{b\,1} \end{array}} \qquad \frac{F_2 Q\exists\exists\ x^{(3)}P^{(3)}x^{(3)}}{\begin{array}{c|c|c} F_2 P^{x\,1}_{a\,1} & T_1 P^{x\,1}_{a\,1} & F_1 P^{x\,1}_{a\,1} \\ F_2 P^{x\,1}_{b\,1} & F_1 P^{x\,1}_{b\,1} & T_1 P^{x\,1}_{b\,1} \end{array}}$$

$$\frac{T_3 Q\exists\exists\ x^{(3)}P^{(3)}x^{(3)}}{\begin{array}{c|c} T_3 P^{x\,1}_{a\,1} & F_1 P^{x\,1}_{a\,1} \\ T_3 P^{x\,1}_{b\,1} & F_1 P^{x\,1}_{b\,1} \end{array}} \qquad \frac{F_3 Q\exists\exists\ x^{(3)}P^{(3)}x^{(3)}}{\begin{array}{c|c|c} F_3 P^{x\,1}_{a\,1} & T_1 P^{x\,1}_{a\,1} & F_1 P^{x\,1}_{a\,1} \\ F_3 P^{x\,1}_{b\,1} & F_1 P^{x\,1}_{b\,1} & T_1 P^{x\,1}_{b\,1} \end{array}}$$

| | | |
|---|---|---|
| $\pi_{1.1}: \forall(x)P(x)$ <br> $T_{1.1}P^x_a \mid F_{1.1}P^x_a$ <br> $T_{1.1}P^x_b \mid F_{1.1}P^x_b$ | $\pi_{2.1}: \exists(x)P(x)$ <br> $T_{2.1}P^x_a \mid F_{2.1}P^x_a$ <br> $F_{2.1}P^x_b \mid T_{2.1}P^x_b$ | $\pi_{3.1}: \exists(x)P(x)$ <br> $T_{3.1}P^x_a \mid F_{3.1}P^x_a$ <br> $F_{3.1}P^x_b \mid T_{3.1}P^x_b$ |
| – | $\pi_{2.2}: \exists(x)P(x)$ <br> $T_{2.2}P^x_a \mid F_{2.2}P^x_a$ <br> $T_{2.2}P^x_b \mid F_{2.2}P^x_b$ | – |
| – | – | $\pi_{3.3}: \exists(x)P(x)$ <br> $T_{3.3}P^x_a \mid F_{3.3}P^x_a$ <br> $T_{3.3}P^x_b \mid F_{3.3}P^x_b$ |

**Duality and self – duality for quantifiers**

Duality

$$\forall\exists\exists \overset{1}{\longleftrightarrow} \exists\exists\exists \overset{2}{\longleftrightarrow} \exists\forall\exists$$

$$\updownarrow 1 \qquad \updownarrow 5 \qquad \updownarrow 1$$

$$\exists\forall\forall \overset{1}{\longleftrightarrow} \forall\forall\forall \overset{2}{\longleftrightarrow} \forall\exists\forall$$

Self – duality

$$\forall\forall\exists \xleftrightarrow{1,2,5} \exists\exists\forall$$

### 3.1.3. Identitive disseminations

**Example** : $MED\left([Log], \langle \wedge \rangle\right)^{(3)}$ :

$id^{(3,1)} : (\wedge \wedge \wedge) : Log \longrightarrow Log.$

$\left(\mathcal{U}_1 \bigcap_{1.2} \mathcal{U}_2\right) \bigcap_{2.3} \mathcal{U}_3 = \emptyset$

$\mathcal{U}^{(3.2)} = \left(\mathcal{U}_{1.1} \amalg_{1.2} \mathcal{U}_{2.2}\right) \amalg_{1.2.3} \mathcal{U}_{3.3}$ :

$\mathcal{U}_{i.j} = \left\{[Log]_{i.j}, [\wedge]_{i.j}\right\}, \; i = j = 1, 2, 3$

$cod(X1) = dom(X2),$

$dom(X1) = dom(X3),$

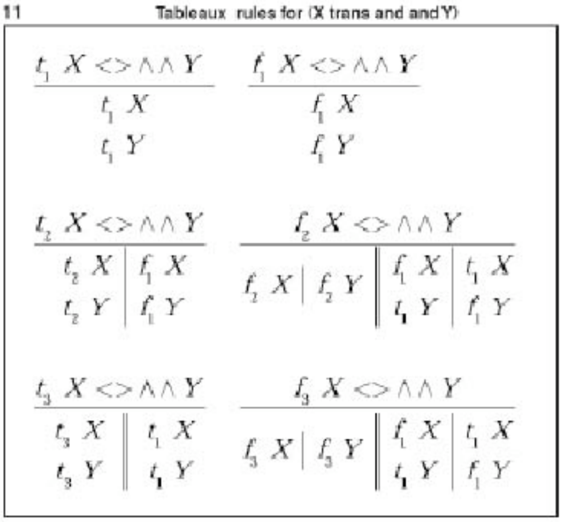
$cod(X2) = cod(X3).$

$$\begin{bmatrix} [B]_1 & [B]_2 & [B]_3 \\ [A]_1 & [A]_2 & [A]_3 \end{bmatrix} :$$

$$\left(\begin{pmatrix} [A]_1 \wedge^{1.0.0} [B]_1 \\ \amalg_{1.2.0} \\ [A]_2 \wedge^{0.2.0} [B]_2 \end{pmatrix} \\ \amalg_{1.2.3} \\ [A]_3 \wedge^{0.0.3} [B]_3 \right) = \left(\begin{pmatrix} [A]_1 \\ \amalg_{1.2.0} \\ [A]_2 \end{pmatrix} \\ \amalg_{1.2.3} \\ [A]_3 \right) \wedge^{1.2.3} \left(\begin{pmatrix} [B]_1 \\ \amalg_{1.2.0} \\ [B]_2 \end{pmatrix} \\ \amalg_{1.2.3} \\ [B]_3 \right)$$

### 3.1.4. Bifurcative disseminations

This paper is offering a sketch of some modi of interactions in a exemplary way as a logical scheme for the interactions of disseminated multiset systems, *dismset*.




A highly explicit formalization of the basic features of the constellation <transjunction, conjunction, conjunction> as given by the logical tableau setting is proposed by the category-theoretic formalization involving *bifunctoriality* to model the distribution of transposition (transjunction) and mediation of the mentioned operators in the contextural grid.
A simplified scheme for (<> ∧ ∧) shows more directly the distribution for transposition, ⋄, and mediation, ∐, in the 3-contextural grid (M, O).

Scheme for [ <> ∧ ∧ ]

**[trans, conj, conj]**

| (M, O) | $O_1$ | $O_2$ | $O_3$ |
|---|---|---|---|
| $M_1$ | 1 – <br> – 2 | – 3 <br> 3 2 | 1 3 <br> 3 – |
| $M_2$ | – | 2 3 <br> 3 3 | – |
| $M_3$ | – | – | 13 <br> 33 |

$$\textbf{val}\left(X <> \wedge\wedge Y\right): \begin{pmatrix} 1 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} = \begin{bmatrix} T_{1.3} & F_{2.3} & F_3 \\ F_{2.3} & F_1 T_2 & F_2 \\ F_3 & F_2 & F_{2.3} \end{bmatrix}:$$

Junctions, $\wedge$, transjunction, $<>$, transposition, $\circ$, mediation, $\amalg$ :
$\left[\left(<> \wedge \wedge\right), \circ, \amalg\right]$.

$$\textbf{true}\left(X <> \wedge\wedge Y\right):$$

$$\left(\begin{pmatrix} t_1\left(\left[A_1\right] <>_{1.1} \left[B_1\right]\right) \\ \amalg_{1.2.0} \\ t_2\left(\left[A_2\right] \wedge_{2.2} \left[B_2\right]\right) \circ_{2.1} f_1\left(\left[A_1\right] \wedge_{2.1} \left[B_1\right]\right) \\ \amalg_{1.2.3} \\ t_3\left(\left[A_3\right] \wedge_{3.3} \left[B_3\right]\right) \circ_{3.1} t_1\left(\left[A_1\right] \wedge_{3.1} \left[B_1\right]\right) \end{pmatrix}\right) =$$

$$\left(\begin{pmatrix} t_1\left[A_1\right] \\ \amalg_{1.2.0} \\ t_2\left[A_2\right] \circ_{2.1} f_1\left[A_1\right] \\ \amalg_{1.2.3} \\ t_3\left[A_3\right] \circ_{3.1} t_1\left[A_1\right] \end{pmatrix}\right) \begin{bmatrix} \wedge_{1.1} & - & - \\ \wedge_{2.1} & \wedge_{2.2} & - \\ \wedge_{3.1} & - & \wedge_{3.3} \end{bmatrix} \left(\begin{pmatrix} t_1\left[B_1\right] \\ \amalg_{1.2.0} \\ t_2\left[B_2\right] \circ_{2.1} f_1\left[B_1\right] \\ \amalg_{1.2.3} \\ t_3\left[B_3\right] \circ_{3.1} t_1\left[B_1\right] \end{pmatrix}\right)$$

$$\textbf{false}\left(X <> \wedge\wedge Y\right):$$

$$\left(\begin{pmatrix} f_1\left(\left[A_1\right] <>_{1.1} \left[B_1\right]\right) \\ \amalg_{1.2.0} \\ f_2\left(\left[A_2\right] \wedge_{2.2} \left[B_2\right]\right) \circ_{2.1} \left(\left(f_1\left[A_1\right] \wedge_{2.1} t_1\left[B_1\right]\right) \vee_{2.1} \left(t_1\left[A_1\right] \wedge_{2.1} f_1\left[B_1\right]\right)\right) \\ \amalg_{1.2.3} \\ f_3\left(\left[A_3\right] \wedge_{3.3} \left[B_3\right]\right) \circ_{3.1} \left(\left(f_1\left[A_1\right] \wedge_{3.1} t_1\left[B_1\right]\right) \vee_{3.1} \left(t_1\left[A_1\right] \wedge_{2.1} f_1\left[B_1\right]\right)\right) \end{pmatrix}\right) =$$

$$\left(\begin{pmatrix} f_1\left[A_1\right] \\ \amalg_{1.2.0} \\ f_2\left[A_2\right] \circ_{2.1} \left(f_1\left[A_1\right] \vee_{2.1} t_1\left[B_1\right] \; 1\right) \\ \amalg_{1.2.3} \\ f_3\left[A_3\right] \circ_{3.1} \left(f_1\left[A_1\right] \vee_{3.1} t_1\left[B_1\right] \; 1\right) \end{pmatrix}\right) \begin{bmatrix} \wedge_{1.1} \\ \wedge_{2.1} \vee_{2.2} \\ \wedge_{3.1} \vee_{3.3} \end{bmatrix} \left(\begin{pmatrix} f_1\left[B_1\right] \\ \amalg_{1.2.0} \\ f_2\left[B_2\right] \circ_{2.1} \left(t_1\left[A_1\right] \vee_{2.1} f_1\left[B_1\right]\right) \\ \amalg_{1.2.3} \\ f_3\left[B_3\right] \circ_{3.1} \left(t_1\left[A_1\right] \vee_{3.1} f_1\left[B_1\right] \; 1 \right.\end{pmatrix}\right.$$

## 3.2. Dissemination of multiset systems

### 3.2.1. Junctional operations

On the base of the sketched polycontextural schemes, disseminations of multiset systems that are containing msets are naturally constructed. Junctional operations of identive mappings are modeling a kind of parallelism and concurrency without interactiviy and reflectionality between disseminated mset systems. Therefore, there are no special conflicts to consider for a consistent formalization as for a modeling of transjunctional, i.e. interactional *mset* systems.

**Example : union** $\bigcup\bigcup\bigcup_{deutero}$

$A = [a, b]_{4,5},\ B = [a, b]_{2,3}$

$$(id,\ id,\ id) : \underset{type,type,type}{\bigcup\bigcup\bigcup} : \left(A^{(3,3)}, B^{(3,3)}\right) \longrightarrow \left(\begin{array}{c|c|c} \bigcup_{1.1} & - & - \\ \hline - & \bigcup_{2.2} & - \\ \hline - & - & \bigcup_{3.3} \end{array}\right) :$$

$$A \bigcup_{id} B = [a, b]_{4,5} \bigcup_{d} \begin{pmatrix} \mu_1 B = [a, b]_{2,3} \\ \mu_2 B = [b, c]_{2,3} \\ \mu_2 B = [c, d]_{2,3} \end{pmatrix} = \begin{pmatrix} [a, b]_{4,5} \\ [a, b, c]_{4,5,3} \\ [a, b, c, d]_{4,5,2,3} \end{pmatrix}.$$

$$\left(\begin{array}{c|c|c} \begin{pmatrix} [a, b]_{4,5} \\ [a, b, c]_{4,5,3} \\ [a, b, c, d]_{4,5,2,3} \end{pmatrix}_{1.1} & - & - \\ \hline - & \begin{pmatrix} [a, b]_{4,5} \\ [a, b, c]_{4,5,3} \\ [a, b, c, d]_{4,5,2,3} \end{pmatrix}_{2.2} & - \\ \hline - & - & \begin{pmatrix} [a, b]_{4,5} \\ [a, b, c]_{4,5,3} \\ [a, b, c, d]_{4,5,2,3} \end{pmatrix}_{3.3} \end{array}\right).$$

### 3.2.2. De Morgan for 3-junctional operations

**Example : union** $\bigcup\bigcup\bigcup_{deutero}$

$$(-A) \cap (-B) = -(A \cup B)$$

$$[a, b, c]_{-1^{2}-2^{3}-3^{1}} \bigcap [a, b]_{-1^{1}-2^{1}} = -\left([a, b, c]_{1^{2}2^{3}3^{1}} \bigcup [a, b]_{1^{1}2^{1}}\right)$$

$$[a, b, c]_{-1^{2}-2^{3}-3^{1}} = -\left([a, b, c]_{1^{2}2^{3}3^{1}}\right) = [a, b, c]_{-1^{2}-2^{3}-3^{1}},$$

$$-((-A) \cap (-B)) = (A \cup B)$$

$$[a, b, c]_{1^{2}2^{3}3^{1}} = [a, b, c]_{1^{2}2^{3}3^{1}}.$$

$A = [a, b]_{4,5},\ B = [a, b]_{2,3}$ :

$$\left(-_1 A^{(3)}\right) \bigcap_1 \bigcap_2 \bigcap_3 \left(-_1 B^{(3)}\right) = -_1\left(A^{(3)} \bigcup_1 \bigcap_3 \bigcap_2 B^{(3)}\right),$$

$$\left(-_2 A^{(3)}\right) \bigcap_1 \bigcap_2 \bigcap_3 \left(-_2 B^{(3)}\right) = -_2\left(A^{(3)} \bigcap_3 \bigcup_2 \bigcap_1 B^{(3)}\right),$$

$$\left(-_3 A^{(3)}\right) \bigcap_1 \bigcap_2 \bigcap_3 \left(-_3 B^{(3)}\right) = -_3\left(A^{(3)} \bigcup_2 \bigcup_1 \bigcup_3 B^{(3)}\right).$$

### 3.2.3. Transjunctional operations

Transjunctional operations on disseminated mappings are modeling a kind of interactivity and interpenetration without reflectionality between disseminated contextural systems.

**Scheme for transjunction : Example :** $\underset{type,type,type}{\uplus \cup \cap}$

$$(bif,\ id,\ id) : \uplus\bigcup\bigcap \longrightarrow \uplus\bigcup\bigcap :$$

$$bif_1 : \uplus\bigcup\bigcap \longrightarrow \uplus_{1.1} \mid \bigcup_{2.1} \mid \bigcap_{3.1} : type$$

$$id_2 : \bigcup \longrightarrow \bigcup : \text{type}$$
$$id_3 : \bigcap \longrightarrow \bigcap : \text{type}$$

$$(\text{bif, id, id}) : \underset{\text{type,type,type}}{\biguplus \bigcup \bigcap} : \left(A^{(3,3)}, B^{(3,3)}\right) \longrightarrow \left(\begin{array}{c|c|c} \biguplus_{1.1} & \bigcup_{2.1} & \bigcap_{3.1} \\ \hline - & \bigcup_{2.2} & - \\ \hline - & - & \bigcap_{3.3} \end{array}\right) :$$

$$\left[\text{bif, id, id}\right] : A^{(3,3)} \underset{\text{type,type,type}}{\biguplus \bigcup \bigcap}^{(3,3)} B^{(3,3)},$$
$$\text{type} < 1.1., 2.1., 3.1;\ 2.2,\ 3.3 > :$$
$$\begin{array}{c} \left(A^{1.1} \biguplus B^{1.1}\right) \diamond \left(A^{2.1} \bigcup B^{2.1}\right) \diamond \left(A^{3.1} \bigcap B^{3.1}\right) \\ \amalg \\ \left(A^{2.2} \bigcup B^{2.2}\right) \\ \amalg \\ \left(A^{3.3} \bigcap B^{3.3}\right) \end{array}$$

### 3.2.4. Reflectional operations

$$(\text{refl, id, id}) : \biguplus \bigcup \bigcap \longrightarrow \biguplus \bigcup \bigcap :$$
$$refl_1 : \biguplus \bigcup \bigcap \longrightarrow \biguplus_{1.1} \mid \bigcup_{1.2} \mid \bigcap_{1.2} : \text{type}$$
$$id_2 : \bigcup \longrightarrow \bigcup : \text{type}$$
$$id_3 : \bigcap \longrightarrow \bigcap : \text{type}$$

$$(\text{refl, id, id}) : \underset{\text{type,type,type}}{\biguplus \bigcup \bigcap} : \left(A^{(3,3)}, B^{(3,3)}\right) \longrightarrow \left(\begin{array}{c|c|c} \biguplus_{1.1} & - & - \\ \hline \bigcup_{1.2} & \bigcup_{2.2} & - \\ \hline \bigcap_{1.3} & - & \bigcap_{3.3} \end{array}\right) :$$

$$\left[\text{refl, id, id}\right] : A^{(3,3)} \underset{\text{type,type,type}}{\biguplus \bigcup \bigcap}^{(3,3)} B^{(3,3)},$$
$$\text{type} < 1.1., 1.2., 1.3;\ 2.2,\ 3.3 > :$$
$$\begin{array}{c} \left(A^{1.1} \biguplus B^{1.1}\right) \\ \amalg \\ \left(A^{1.2} \bigcup B^{1.2}\right) \square \left(A^{2.2} \bigcup B^{2.2}\right) \\ \amalg \\ \left(A^{1.3} \bigcap B^{1.3}\right) \square \left(A^{3.3} \bigcap B^{3.3}\right) \end{array}$$

### 3.2.5. Dissemination of mixed abstract data types

It is supposed that mostly a complexion of processes, especially in living systems, cannot be adequately modeled with one abstract data type alone. A complexion of abstract data types that are supporting more concrete data types are necessary, all interacting at once together. Such a situation has no correct modeling and formalization in mono-contextural logics and systems theory. Earlier sketches of a general framework of complex programming and programming the complexity of living systems had been published as "*Contextural Programming*" at ConTxTures:
http://www.thinkartlab.com/pkl/lola/ConTeXtures.pdf

The example A $\diamond \cup \cup$ $B_{trito, mset, set}$ below shows an interpenetration of $subsystem_2$ and $subsystem_3$ onto $subsystem_1$.
Specifically, the $subsystem_1$ wit its ‘data type' "*tritogram*" gets an interpenetration by transjunction from $subsystem_2$ with its ‘data type' "*msets*" and from $subsystem_3$ with its ‘data type' "*sets*". All this together holds simultaneously. Junctional situations that are holding or running separately in parallel are covered by $subsystem_2$ as such with its ‘data type' "*mset*" and by the $subsystem_3$ as such with its ‘data type' "*set*".

The logical constellation of this simultaneity of different abstract data types at a ‘single' contextural locus is ruled by the logical tableaux for transjunctional quantification. The proposed formula for A $\diamond \cup \cup$ $B_{trito, mset, set}$ is not yet taking this logical constellation manifestly into account in its presentation. The logical background becomes manifest in the further use and development of the formula. A fact not unknown in set or multiset theory that are based on classical logic.

**Team observation**
With the dissemination of objects over the kenomic matrix it becomes obvious that an adequate observation of the events in the complexion is not realizable by a single external observer with a single or multiple observations but needs a *team* of observers with an adequate team structure to be observed, modeled and formalized.

***Junctional dissemination***

**Example: union** $\bigcup\bigcup\bigcup_{set,\, mset,\, mset}$

$A^1 = \{a, b\}_{1,1}$ , $A^2 = [a, b]_{2,3}$ , $A^3 = [a, b]_{2,3}$

$B^1 = \{a, b\}_{1,1}$ , $B^2 = [a, b]_{2,3}$ , $B^3 = [a, b]_{2,3}$

$A \bigcup\bigcup\bigcup_{set,\, mset,\, mset} B =$

$$\begin{pmatrix} \{a, b\}_{1,1} \\ [a, b]_{2.3} \\ [a, b]_{2,3} \end{pmatrix} \bigcup\bigcup\bigcup_{set,\, mset,\, mset} \begin{pmatrix} \{a, b\}_{1,1} \\ [a, b]_{2.3} \\ [a, b]_{2,3} \end{pmatrix} = \begin{pmatrix} \{a, b\}_{1,1} \\ [a, b]_{4,6} \\ [a, b]_{4,6} \end{pmatrix}.$$

**Example: union** $\bigcup\bigcup\bigcup_{set,\, mset,\, mset}$

$A^1 = \{a, b\}_{1,1}$ , $A^2 = [a, b]_{2,3}$ , $A^3 = [a, b]_{2,3}$

$B^1 = \{a, b\}_{1,1}$ , $B^2 = [b, c]_{2,3}$ , $B^3 = [c, d]_{2,3}$

$A \bigcup\bigcup\bigcup_{set,\, mset,\, mset} B =$

$$\begin{pmatrix} \{a, b\}_{1,1} \\ [a, b]_{2.3} \\ [a, b]_{2,3} \end{pmatrix} \bigcup\bigcup\bigcup_{set,\, mset,\, mset} \begin{pmatrix} \{a, b\}_{1,1} \\ [b, c]_{2.3} \\ [c, d]_{2,3} \end{pmatrix} = \begin{pmatrix} \{a, b\}_{1,1} \\ [a, b, c]_{2,5,3} \\ [a, b, c, d]_{2,3,2,3} \end{pmatrix}$$

Null

**Example : union** $\bigcup\bigcup\bigcup_{set,\ mset,\ trito}$

$A^1 = \{a, b\}_{1,1}$ , $A^2 = [a, b]_{1,1}$ , $A^3 = [a, b]_{1^2\,2^1}$

$B^1 = \{a, b\}_{1,1}$ , $B^2 = [b, c]_{1,1}$ , $B^3 = [a, b]_{1^2\,2^1}$

$$\bigcup\bigcup\bigcup_{set,\ mset,\ trito} = \begin{pmatrix} \{a, b\}_{1,1} \\ [a, b]_{2.3} \\ ([a, b]_{1^2\,2^1}) \end{pmatrix} \bigcup\bigcup\bigcup_{set,\ mset,\ trito} \begin{pmatrix} \{a, b\}_{1,1} \\ [b, c]_{2.3} \\ \begin{pmatrix} [a, b]_{1^2\,2^1} \\ [a, c]_{1^2\,3^1} \\ [b, a]_{2^2\,1^1} \\ [b, c]_{2^2\,3^1} \\ [c, a]_{3^2\,1^1} \\ [c, b]_{3^2\,2^1} \\ [c, d]_{3^2\,4^1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \{a, b\}_{1,1} \\ [a, b, c]_{2,3,3} \\ \begin{pmatrix} [a, b]_{1^2\,2^1} \\ [a, b, c]_{1^2\,2^1\,3^1} \\ [a, b]_{1^2\,2^2} \\ [a, b, c]_{1^2\,2^2\,3^1} \\ [a, b, c]_{1^2\,2^1\,3^2\,4^1} \\ [a, b, c]_{1^2\,2^1\,3^2\,4^1} \\ [a, b, c, d]_{1^2\,2^1\,3^2\,4^1} \end{pmatrix} \end{pmatrix}$$

**Example : union** $\bigcup\bigcup\biguplus_{set,\ mset,\ trito}$

$A^1 = \{a, b\}_{1,1}$ , $A^2 = [a, b]_{1,1}$ , $A^3 = [a, b]_{1^2\,2^1}$

$B^1 = \{a, b\}_{1,1}$ , $B^2 = [b, c]_{1,1}$ , $B^3 = [a, b]_{1^2\,2^1}$

$$A \bigcup\bigcup\biguplus_{set,\ mset,\ trito} B =$$

$$\begin{pmatrix} \{a, b\}_{1,1} \\ [a, b]_{2.3} \\ ([a, b]_{1^2\,2^1}) \end{pmatrix} \bigcup\bigcup\biguplus_{set,\ mset,\ trito} \begin{pmatrix} \{a, b\}_{1,1} \\ [b, c]_{2.3} \\ \begin{pmatrix} [a, b]_{1^2\,2^1} \\ [a, c]_{1^2\,3^1} \\ [b, a]_{2^2\,1^1} \\ [b, c]_{2^2\,3^1} \\ [c, a]_{3^2\,1^1} \\ [c, b]_{3^2\,2^1} \\ [c, d]_{3^2\,4^1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \{a, b\}_{1,1} \\ [a, b, c]_{2,3,3} \\ \begin{pmatrix} [a, b]_{1^2\,2^1\,1^2\,2^1} \\ [a, b, c]_{1^2\,2^1\,1^2\,3^1} \\ [a, b]_{1^2\,2^3\,1^1} \\ [a, b, c]_{1^2\,2^3\,3^1} \\ [a, b, c, d]_{1^2\,2^1\,3^2\,1^1} \\ [a, b, c, d]_{1^2\,2^1\,3^2\,2^1} \\ [a, b, c, d]_{1^2\,2^1\,3^2\,3^1} \end{pmatrix} \end{pmatrix}.$$

### *Transjunctional dissemination*

**Scheme for transjunction** $\diamond$ : $\biguplus\bigcup\bigcup_{trito,\ mset,\ set}$

$$(bif,\ id,\ id) : \biguplus\bigcup\bigcup_{trito,\ mset,\ set} \longrightarrow \biguplus\bigcup\bigcup_{trito,\ mset,\ set} :$$

$$bif_1 : \biguplus\bigcup\bigcup_{trito,\ mset,\ set} \longrightarrow \biguplus_{trito}^{1.1} \Big| \bigcup_{mset}^{2.1} \Big| \bigcup_{mset}^{3.1}$$

$$\text{id}_2 : \underset{\text{trito, mset, set}}{\biguplus \bigcup \bigcup} \longrightarrow \underset{\text{mset}}{\bigcup}$$

$$\text{id}_3 : \underset{\text{trito, mset, set}}{\biguplus \bigcup \bigcup} \longrightarrow \underset{\text{set}}{\bigcup}$$

$$(\text{bif, id, id}) : \underset{\text{trito, mset, set}}{\biguplus \bigcup \bigcup} : \left(A^{(3,3)}, B^{(3,3)}\right) \longrightarrow \left(\begin{array}{c|c|c} \underset{\text{trito}}{\biguplus}\ 1.1 & \underset{\text{mset}}{\bigcup}\ 2.1 & \underset{\text{set}}{\bigcup}\ 3.1 \\ \hline - & \underset{\text{mset}}{\bigcup}\ 2.2 & - \\ \hline - & - & \underset{\text{set}}{\bigcup}\ 3.3 \end{array}\right) :$$

$$(\text{bif, id, id}) : A^{(3)} \underset{\text{trito, mset, set}}{\biguplus \bigcup \bigcup} B^{(3)} =$$

type < 1.1., 2.1., 3.1; 2.2, 3.3 > :

$$\left(A^{1.1} \underset{\text{trito}}{\biguplus}_{1.1} B^{1.1}\right) \diamond_{2.1} \left(A^{2.1} \underset{\text{mset}}{\bigcup} B^{2.1}\right) \diamond_{3.1} \left(A^{3.1} \underset{\text{set}}{\bigcup} B^{3.1}\right)$$

$$\amalg$$

$$\left(A^{2.2} \underset{\text{mset}}{\bigcup} B^{2.2}\right)$$

$$\amalg$$

$$\left(A^{3.3} \underset{\text{set}}{\bigcup} B^{3.3}\right)$$

**Example: transjunction** $[\text{bif, id, id}]$ **:**

$$A \underset{\text{trito, mset, set}}{\biguplus \bigcup \bigcup} B$$

$$A^1 = [a, b]_{1\,2\,2\,1}, \quad A^2 = [a, b]_{2,3}, \quad A^3 = \{a, b\}_{1,1}$$

$$B^1 = [a, b]_{1\,2\,2\,1}, \quad B^2 = [b, c]_{2,3}, \quad B^3 = \{a, b\}_{1,1}$$

$$[\text{bif, id, id}] : A \underset{\text{trito, mset, set}}{\biguplus \bigcup \bigcup} B =$$

$$\left(\begin{array}{l} \left([a, b]_{1\,2\,2\,1} \underset{\text{trito}}{\biguplus}_{1.1} [a, b]_{1\,2\,2\,1}\right) \diamond_{2.1} \\ \left([a, b]_{1\,2\,2\,1} \underset{\text{mset}}{\bigcup}_{2.1} [a, b]_{1\,2\,2\,1}\right) \diamond_{3.1} \\ \left([a, b]_{1\,2\,2\,1} \underset{\text{set}}{\bigcup}_{3.1} [a, b]_{1\,2\,2\,1}\right) \end{array}\right)$$

$$\amalg_{1.2}$$

$$\left([a, b]_{2,3} \underset{\text{mset}}{\bigcup}_{2.2} [b, c]_{2,3}\right)$$

$$\amalg_{1.2.3}$$

$$\left(\{a, b\}_{1,1} \underset{\text{set}}{\bigcup}_{3.3} \{a, b\}_{1,1}\right).$$

$$\left(\begin{array}{l} [\text{bif, id, id}]:\ A \underset{\text{trito, mset, set}}{\uplus \bigcup \bigcup} B = \\ \\ \textbf{trito}:\ \left([a,b]_{1^2 2^1} \uplus_{1.1} [a,b]_{1^2 2^1}\right) \diamond_{2.1} \\ \textbf{mset}:\ \left([a,b]_{2,3} \bigcup_{2.1} [b,c]_{2.,3}\right) \diamond_{3.1} \\ \textbf{set}:\ \left([a,b]_{1,1} \bigcup_{3.1} [a,b]_{1,1}\right) \\ \hline \textbf{mset}_{2.2}:\ [a,b]_{2,3} \bigcup_{2.2} [b,c]_{2,3} \\ \hline \textbf{set}_{3.3}:\ \{a,b\}_{1,1} \bigcup_{3.3} \{a,b\}_{1,1} \end{array}\right) = \left(\begin{array}{l|l|l} \textbf{tritogram}_{1.1} & \textbf{mset}_{2.1} & \textbf{set}_{3.1} \\ [a,b]_{1^2 2^1 1^2 2^1} & & \\ [a,b,c]_{1^2 2^1 1^2 3^1} & & \\ [a,b]_{1^2 2^3 1^1} & [a,b,c]_{2,5,3} & \{a,b\}_{1,1} \\ [a,b,c]_{1^2 2^3 3^1} & & \\ [a,b,c,d]_{1^2 2^1 3^2 1^1} & & \\ [a,b,c,d]_{1^2 2^1 3^2 2^1} & & \\ [a,b,c,d]_{1^2 2^1 3^2 3^1} & & \\ \hline - & \textbf{mset}_{2.2} & - \\ & [a,b,c]_{2,5,3} & \\ \hline - & - & \textbf{set}_{3.3} \\ & & \{a,b\}_{1,1} \end{array}\right)$$

### 3.2.6. Algebraic properties of disseminated mset-operations

$\textbf{mset} = \left(\text{mset}, \bigcap, \bigcup, \uplus, \diamond, \square\right):$

$\text{algebr}(\textbf{mset}) = \left(\text{mset}, \bigcap, \bigcup, \uplus, \diamond, \text{com, ass, idem, iden, distr, interch}\right)$

$\text{algebr}\left(\text{mset}^{(m,n)}\right) = \text{diss}_{\text{sops}}\left(\text{algebr}(\text{mset}), \text{alg}\right)$

$\text{alg} = \{\text{com, ass, idem, iden, distr}\},$

$\text{sops} = (\text{id, red, iter, refl, perm, bif}\}$

**Monoform**

$A \bigcup \bigcup \bigcup B$

**Heteroform**

$A \bigcup \bigcup \bigcap B$

$A \bigcup \uplus \bigcap B$

**Bifunctorial**

$(A \circ C) \diamond (B \circ D) = (A \diamond B) \circ (C \diamond D)$

$(A \circ C) \square (B \circ D) = (A \square B) \circ (C \square D)$

**Duality**

dual $: \bigcup \longrightarrow \bigcap$

$: \uplus \longrightarrow \backslash$

self – dual $: \diamond \longrightarrow \diamond$ (transposition)

$: \square \longrightarrow \square$ (reflection)

**Bifunctoriality for transposition**

$$\left(\text{Junctors, transjunctors, transposition, mediation}\right) : \; < \left( <> \; \mathbf{v} \; \mathbf{v} \right), \; \diamond, \; \amalg >$$

$$\textbf{true} : A \diamond \bigcup\bigcup B_{\text{trito, mset, set}}\} :$$

$$\left( \begin{pmatrix} t_1 \left(\text{trito}\right) \\ \amalg_{1.2.0} \\ t_2 \left(\text{mset}\right) \; \diamond_{2.1} \; f_1 \left(\text{trito}\right) \\ \amalg_{1.2.3} \\ t_3 \left(\text{set}\right) \; \diamond_{3.1} \; t_1 \left(\text{trito}\right) \end{pmatrix} \right) =$$

$$\left( \begin{pmatrix} t_1 [\text{trito}] \\ \amalg_{1.2.0} \\ t_2 [\text{mset}] \; \diamond_{2.1} \; f_1 [\text{trito}] \\ \amalg_{1.2.3} \\ t_3 [A_3] \; \diamond_{3.1} \; t_1 [\text{trito}] \end{pmatrix} \right) \begin{bmatrix} \diamond_{1.1} & - & - \\ \diamond_{2.1} & \vee_{2.2} & - \\ \diamond_{3.1} & - & \mathbf{v}_{3.3} \end{bmatrix} \left( \begin{pmatrix} t_1 [\text{trito}] \\ \amalg_{1.2.0} \\ t_2 [\text{mset}] \; \diamond_{2.1} \; f_1 [\text{trito}] \\ \amalg_{1.2.3} \\ t_3 [\text{set}] \; \diamond_{3.1} \; t_1 [\text{trito}] \end{pmatrix} \right)$$

$$\textbf{false} : \log\left(A \diamond \bigcup\bigcup B_{\text{trito, mset, set}}\}\right) :$$

$$\left( \begin{pmatrix} f_1 [\text{trito}] \\ \amalg_{1.2.0} \\ f_2 [\text{mset}] \; \diamond_{2.1} \left( \left( f_1 [\text{trito}] \wedge_{2.1} t_1 [\text{trito}] \right) \vee_{2.1} \left( t_1 [\text{trito}] \wedge_{2.1} f_1 [\text{trito}] \right) \right) \\ \amalg_{1.2.3} \\ f_3 [\text{set}] \; \diamond_{3.1} \left( \left( f_1 [\text{trito}] \wedge_{3.1} t_1 [\text{trito}] \right) \vee_{3.1} \left( t_1 [\text{trito}] \wedge_{2.1} f_1 [\text{trito}] \right) \right) \end{pmatrix} \right) =$$

$$\left( \begin{pmatrix} f_1 [\text{trito}] \\ \amalg_{1.2.0} \\ f_2 [\text{mset}] \; \diamond_{2.1} \left( f_1 [\text{trito}] \; \vee_{2.1} t_1 [\text{trito}] \; 1 \right) \\ \amalg_{1.2.3} \\ f_3 [\text{set}] \; \diamond_{3.1} \left( f_1 [\text{trito}] \; \vee_{3.1} t_1 [\text{trito}] \; 1 \right) \end{pmatrix} \right) \begin{bmatrix} \wedge_{1.1} \\ \wedge_{2.1} \wedge_{2.2} \\ \wedge_{3.1} \wedge_{3.3} \end{bmatrix} \left( \begin{pmatrix} f_1 [\text{trito}] \\ \amalg_{1.2.0} \\ f_2 [\text{mset}] \; \diamond_{2.1} \left( t_1 [\text{trito}] \; \vee_{2.1} f_1 \right. \\ \amalg_{1.2.3} \\ f_3 [\text{set}] \; \diamond_{3.1} \left( t_1 [\text{trito}] \; \vee_{3.1} f_1 \right. \end{pmatrix} \right.$$

http://www.thinkartlab.com/pkl/lola/AFOSR-Place-Valued-Logic.pdf
http://www.thinkartlab.com/pkl/lola/From%20Ruby%20to%20Rudy.pdf

# 4. Possible applications of graphematics for Membrane computing

## 4.1. Membrane computing and multisets in hierarchies

[Păun 2005]: *"parallelism a dream of computer science, a common sense in biology"*.

"Any cell means membranes. The cell itself is defined - separated from its environment - by a membrane, the external one. Inside the cell, several membranes enclose "protected reactors", compartments where specific biochemical processes take place. In particular, a membrane encloses the nucleus (of eukaryotic cells), where the genetic material is placed.

"We have mentioned above the notion of a *multiset*. The compartments of a cell contains substances (ions, small molecules, macromolecules) swimming in an aqueous solution; there is no ordering there, everything is close to everything, the concentration matters, the population,

the number of copies of each molecule (of course, we are abstracting/idealizing here, departing from
the biological reality). Thus, the suggestion is immediate: to work with sets of objects whose multiplicities matter, hence with multisets. This is a data structure with peculiar characteristics, not new but not systematically investigated in computer science."
http://psystems.disco.unimib.it/download/MembIntro2004.pdf

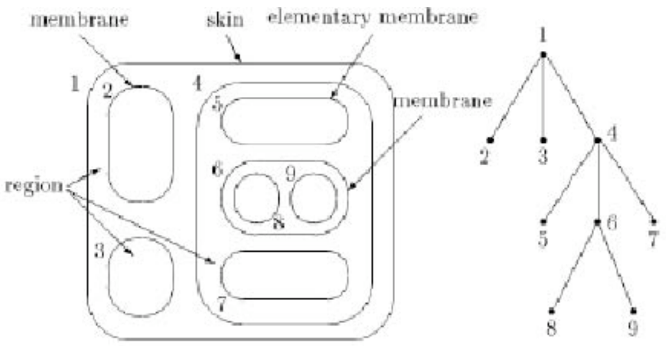



Figure 1: A membrane structure and its associated tree

"More mathematically stated, we look to the set of rules, and try to find a multiset of rules, by assigning multiplicities to rules, with two properties:
(i) the *multiset* of rules is applicable to the multiset of objects available in the respective region, that is, there are enough objects in order to apply the rules a number of times as indicated by their multiplicities, and
(ii) the multiset is maximal, no further rule can be added to it (because of the lack of available objects)."
http://www.macs.hw.ac.uk/~pier/

### 4.2. Multisets in heterarchies

*"Interchangeability not even a dream of computer science, a common interaction in living systems."*

Păun's text gives an impressive plaidoyer for the use of mutisets in computer science and for the modeling of living systems. Despite the highly technical elaborations of P-systems or Membrane systems, it seems that this approach is giving just a snapshot of a living organism but is not thematizing the *dynamic* mechanisms of living matter as such.
Its emphasis is on hierarchical systems, described mainly by the data structure of multisets but neglecting the interchangeability of hierarchies that are establishing *heterarchies* between parts and wholes, or domains and system.
The interchangeability, chiasm or proemiality of the inside/outside mechanism is not in the focus and is strictly excluded by the hierarchy of the general model that is strictly mirrored in the formal apparatus.

## 5. Appendix 1: Elements for programming trito-grammatics

**TRITO-EQUIVALENCE**: a=b, ab=ba, aa!=ab, aab!=aba $\in \Sigma_{tnf}$

### 5.1. Equivalence classes

Tritogram[A] = [abcddc]
Tritogram[B] = [•Oc♣♣c]
dec([A]) = ([a], [b], [c], [dd], [c])
dec([B]) = ([•], [O], [c], [♣♣], [c]).
[A] $=_{trito}$ [B] iff ([a] $=_{ken}$[•], [b] $=_{ken}$[O], [c] $=_{ken}$[c], [dd] $=_{ken}$ [♣♣]).

But this equivalence relation approach is, albeit correct, misleading because it is still too much relying on the identity of its signs.

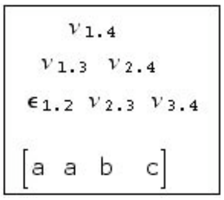
### 5.2. Morphogrammatics

**є/v-structure**
How to define the equivalence of tritograms?

The є/v-approach is checking just the equality (є) or non-equality (v) of objects (signs) of a collection or a string and not the individual atomic signs as such.

[A] = [aabc]

ENstructure[A]




```
datatype EN =E|N;
fun delta (i, j) z=
if  (pos i z) = (pos j z)
    then (i, j, E)
     else (i, j, N) ;
```

```
- ENstructure ["a", "a", "b", "c"];
> [[],
   [(1,2, E),
   [(1, 3, N), (2, 3, N)],
   [(1, 4, N), (2, 4, N), (3, 4, N)]]: enstruct
```

The є/v-analysis of a constellation of objects determines the tritogram [A].trito.
For further elaborations, this є/v-result of the tritogram might be transformed into a sequential form of a keno-sequence (kseq) with the operation ENtoKS.

This is realized by the ML function: **ENtoKS**

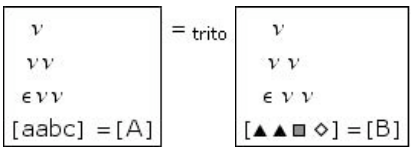
```
- ENtoKS [[],
              [(1,2, E),
              [(1, 3, N), (2, 3, N)],
              [(1, 4, N), (2, 4, N), (3, 4, N)]]
> [1, 1, 2, 3] : kseq

- ENtoKS ENstructure ["b", "a", "d", "c"];
> [1, 2, 3, 4] : kseq
```

**Trito normal form tnf**
A given keno-sequence might not be in a standard normal form (tnf), hence the ML function *tnf*, delivering *ks* shall be applied.

```
tnf: [c,d,d,a] --> [a,b,b,c], kseq.
- val a = [2, 2, 1, 1] : kseq
> val a = [2, 2, 1, 1] kseq
-tnf a;
>  [1, 1, 2, 2] : kseq
```

**Trito-equivalence**
Two tritograms [A] and[B] are trito-equivalent iff their є/v-structures are equal.

[A] $=_{\text{trito}}$ [B] iff EN([A]) = EN([B]).




**In contrast: Equality for multisets**
"Two multisets A and B are **equal**, or the same, if for any object x

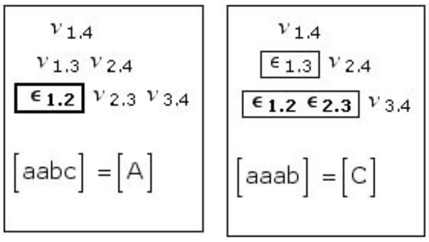
mA (x) = mB (x)." (Sing)

Hence, from the example: [A] $!=_{mset}$ [B] but [A] $=_{tset}$ [B].

### 5.3. Monomorphies

Strings of signs consist of *atomic* signs form an aphabet. This is leading all the following definitions for strings, especially the definition of substitution.

Tritograms and Tritosets are patterns and not strings and consist of *monomorphies*.
The mein properties or operations on singular tritosets are:
primary operations ={tnf, card, lex, num, dec, ken, pos}.




[A] = [aabc]:
[A].dec = [$mg_1$, $mg_2$, $mg_3$]
$mg_{1.1}$.ken = [aa],
$mg_{2.2}$.ken = [b],
$mg_{3.3}$.ken = [c].

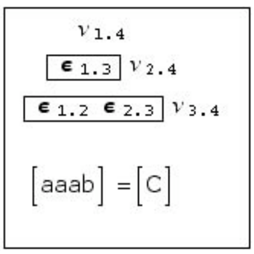
[C] = [aaab]
[C].dec = [$mg_1$, $mg_2$]
[$mg_{1.1}$].ken = [aaa]
[$mg_{2.2}$].ken = [b].

[A].dec: [A].EN = **(1,2, E)**
- ENstructure ["a", "a", "b", "c"];
> [[],
  [**(1,2, E)**,
  [(1, 3, N), (2, 3, N)],
  [(1, 4, N), (2, 4, N), (3, 4, N)]]: enstruct.

[A].dec = [$mg_1$,$mg_2$, $mg_3$]

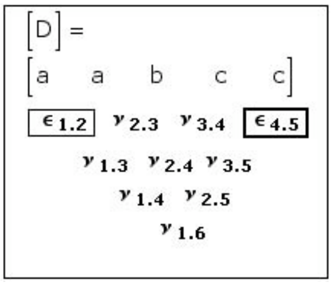
[C].dec: [C].EN = (1,2, **E**), (2,3, **E**)
- ENstructure ["a", "a", "a", "b"];
> [[],
  [(1,2, **E**),
  [(1, 3, **E**), (**2, 3, E**)],
  [(1, 4, N), (2, 4, N), (3, 4, N)]]: enstruct.

Construction of monomorphies from ENstructure




[C].dec = [(**1,2, E**) ∪ (**2,3, E)** ∪ (**1,3, E)**] => mg:[$mg_1$, $mg_2$]
$mg_1$ = ((1,2)+(2,3)+(1,3), E)
[C].dec = [$mg_1$, $mg_2$]
ken([$mg_1$]) = [aaa],

ken[$mg_2$] = [b].
*Numeric functions*
card([$mg_1$] = 3
lex([$mg_1$]) = 1
num([$mg_1$]) = (lex$^{card}$([$mg_1$]) = $1^3$.
num([$mg_2$]) = $2^1$
num([MG]) = num([$mg_1$]), num([$mg_2$]),
num([MG]) = $1^3 2^1$.




- ENstructure ["a", "a", "b", "c", "c"];
> [[],
  [(**1,2, E**),
  [(1, 3, N), (2, 3, N) ],
  [(1, 4, N), (2, 4, N), (3, 4, N)],
  [(1, 6, N), (2, 5, N), (3, 5, N), (**4,5,E**]: enstruct.

[D].dec = [ [(**1,2, E**) ∩ (**4,5,E**] = ∅] => mg:[$mg_1$, $mg_2$, $mg_3$]

[D].dec = [$mg_1$, $mg_2$, $mg_3$]
ken([$mg_1$]) = [aa],
ken[$mg_2$] = [b],
ken[$mg_3$] = [cc].

**ML procedures in: Morphogrammatik, p. 46, 49-52**
http://works.bepress.com/thinkartlab/15/

## 5.4. Math of Monomorphies

Let A and B nonempty finite sets A = {$a_1$, $a_2$, ..., $a_n$} and B = {$b_1$, $b_2$, ..., $b_m$}
Let $B^A$ denote the set of allmappings from A to B,

$B^A$ ={μ | μ: A --> B}.

This is elaborated at: *Morphogrammatik*.

**How to construct monomorphies mathematically?**
The question: *What replaces atomic signs in a kenogrammatic pattern (morphogram)?* Is answered by Schadach with the introduction of *monomorphies* of morphograms.

From a mathematical point of view, monomorphies are *partitions* of mappings. This is well elaborated by [Schadach 1967]. The procedure to build monomorphies out from morphograms, as it is mathematically defined by Schadach's approach, shall be called *monomorphic decomposition*.

" Let A and B be non - empty finite sets,

$A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, b_2, \ldots, b_m\}$.

Let denote $B^A$ the set of all mappings from A to B, $B^A = \{\mu \mid \mu : A \longrightarrow B\}$, card $B^A$ = (card B) ca

The following theorem shows that
every family of subsets $B^A$ defines a certain *partition* of $B^A$.

**Theorem 1.**

Let $\{R_i \mid i \in I\}$ be a family of subsets of $B^A$ where I is a finite index set; $R_i \subseteq B^A$ for eacl

The family $\{R_i \mid i \in I\}$ defines a partition of $B^A$ such that
the elements of the partition (the equivalence classes of mappings) are

$$[\mu]_{I_x} = \bigcap_{i_x \in I_x} R_{i_x} - \bigcup_{i_y \in I - I_y} R_{i_y}$$

where $I_x$ runs through all subsets of I.

**Corollary 1.**

If $I_x = \emptyset$, then $[\mu]_\emptyset = B^A - \bigcup_{i \in I} R_i$ and

if $I_x = I$, then $[\mu]_I = \bigcap_{i \in I} R_i$.

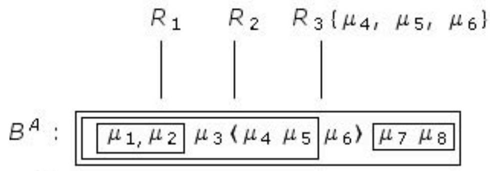
**Corollary 2**

By Theorem 1, we get *a* mapping from the set of all families of subsets of $B^A$ onto the set of all partitions of $B^A$.

**Example.**

Let be $B^A = \{\mu_1, \mu_2 \ldots, \mu_8\}$ and the family of subsets $\{R_i \mid i \in I = \{1, 2, 3\}\}$ where

$R_1 = \{\mu_1, \mu_2\}$,
$R_2 = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$,
$R_3 = \{\mu_4, \mu_5, \mu_6\}$.




Null

| $I_x$ | $[\mu]_{I_x}$ |
|---|---|
| $\emptyset$ | $B^A - (R_1 \smile R_2 \smile R_3) = \{\mu_7, \mu_8\}$ |
| $\{1\}$ | $R_1 - (R_2 \smile R_3) = \emptyset$ |
| $\{2\}$ | $R_2 - (R_1 \smile R_3) = \{\mu_3\}$ |
| $\{3\}$ | $R_3 - (R_1 \smile R_2) = \{\mu_6\}$ |
| $\{1, 2\}$ | $(R_1 \frown R_2) - R_3) = \{\mu_1, \mu_2\}$ |
| $\{1, 3\}$ | $(R_1 \frown R_3) - R_2 = \{\mu_7, \mu_8\}$ |
| $\{2, 3\}$ | $(R_2 \smile R_3) - R_1 = \{\mu_4, \mu_5\}$ |
| $I$ | $R_1 \frown R_2 \frown R_3 = \emptyset$ |

$$B^A : \boxed{\boxed{\mu_1, \mu_2}\ \boxed{\mu_3}\ \boxed{\mu_4\ \mu_5}\ \boxed{\mu_6}\ \boxed{\mu_7\ \mu_8}}.$$

(Dieter J. Schadach, BCL Report No. 4.1, August 1, 1967)